Abundance of one dimensional non uniformly hyperbolic attractors for surface endomorphisms

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Abstract

For every C^2-small perturbation B, we prove that the map $(x, y) \mapsto (x^2 + a + 2y, 0) + B(x, y)$ preserves a physical, SRB probability, for a Lebesgue positive set of parameters a. When the perturbation B is zero, this is the Jackobson theorem; when the perturbation is a small constant times $(0, 1)$, this is the celebrated Benedicks-Carleson theorem.

In particular, a new proof of the last theorems is given, by basically mixing analytical ideas of Benedicks-Carleson and the combinatorial formalism of Yoccoz puzzle with new geometrical and algebraic ingredients. These proofs are enlarged to the C^2-topology. Also the dynamics can be an endomorphism.

Our aim is to prove the existence of a non-uniformly hyperbolic attractor for a large set of parameters $a \in \mathbb{R}$, for the following family of maps:

$$f_{a,B} : \ (x, y) \mapsto (x^2 + a + 2y, 0) + B(x, y),$$

where B is a fixed C^2-map of $\mathbb{R}^2$ close to 0. We denote by b an upper bound of the uniform C^2-norm of B and of the Jacobian of $f_{a,B}$. For B fixed, we prove that for a large set Ω_B of parameters a, the dynamics $f_{a,B}$ is *strongly regular*. This ought to imply a lot of properties. Belong those, a simple consequence is the following theorem.

Theorem 0.1. *For any $t \in (0, 1)$, there exist a_0 greater but close to -2 and $b > 0$ such that for any B with C^2-norm less than b, there exists a subset $\Omega_B \subset [-2, a_0]$ of relative measure greater than t satisfying the following property. For any $a \in \Omega_B$, there exists an union of unstable manifolds A which support a unique SRB measure which is ergodic and whose basin has positive Lebesgue measure.*

This answers to a question of Pesin-Yurchenko for reaction-diffusion PDEs in applied mathematics [PY04].

1 Introduction

The birth of chaotic dynamical system goes back to Poincaré in his study of the 3-body problem. Let us show a simplification of his counterexample about the former belief that dynamical systems are deterministic: small perturbations do not change the long term behavior. The idea is to consider in the plane $\mathbb{C}$ two heavy planets of equal mass and in circular orbits around 0. Then make the product of this plane with the circle $\mathbb{S}$ to obtain the cylinder $\mathbb{C} \times \mathbb{S}$. Put a planet P with negligible mass at the antipodal point $1 \in \mathbb{S}$ of 0, and with a vertical initial speed v. As the planet has negligible mass the motions of both heavy planets remain circular. On the other hand, the dynamics of the planet remains in the circle $\{0\} \times \mathbb{S}$ and is equal to the pendulum dynamics f of the tangent space of the circle $T\mathbb{S} = \mathbb{S} \times \mathbb{R}$. We identify $T\mathbb{S}$ to the punctured plan in the phase diagram drawn at the left of fig. 1. It turns out that the points $M = (1, 0) \in T\mathbb{S}$ is a hyperbolic fixed point: the differential of f at M has two eigenvalues of modulus different from 1. Also the stable and unstable manifolds of M are equal to a same curve $W^s(M) = W^u(M)$. We call this a homoclinic tangency. Contrarily to the pendulum, we can perturb the system such that $W^s(M) \neq W^u(M)$ (see the second picture of fig. 1). For we can consider that the two heavy planets have an elliptic orbit centered at 0 with small eccentricity $e \neq 0$. The hyperbolic point M persists but its stable and unstable manifolds $W^s(M)$ and $W^u(M)$ are not anymore equal. The intersection of $W^s(M)$ with $W^u(M)$ is called a homoclinic intersection.

The complex drawn picture of $W^s(M)$ and $W^u(M)$ makes at itself the birth of the chaotic dynamical systems (see third picture of fig. 1).

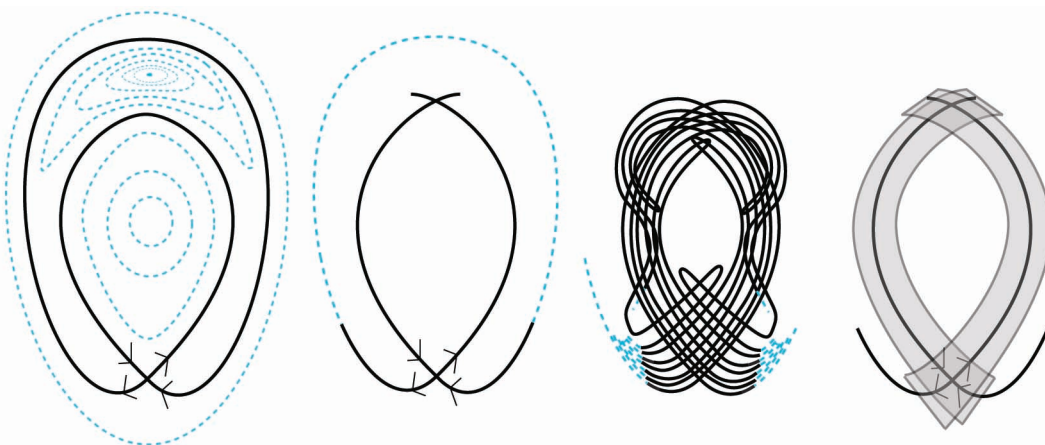Nowadays, we do not know how to describe this picture mathematically, for many perturbations.

Figure 1: Homoclinicle intersection arising from a 3-body problem.

In the sixties, Smale remarked that a thick neighborhood of a local stable manifold of M is sent to a thick neighborhood $\mathcal{H}$ of an unstable manifold of M, both containing a same point of the homoclinic intersection not equal to M (see the last picture of fig. 1). For a satisfactory

neighborhood $\mathcal{H}$, the maximal invariant $K = \cap_{n \in \mathbb{Z}} f^n(\mathcal{H})$ is a hyperbolic compact set: the celebrated Smale Horseshoe. This means that the space K is endowed with two Tf-invariant directions, one exponentially expanded, the other exponentially contracted by the dynamics.

With hyperbolic dynamical system, an optimistic wave in dynamics was born ([Sma67]). Together with Bowen, Palis, Sinai and others, many properties of these compact sets were shown: structural stability, description of the geometry of these sets, and for the attractors, the existence of Markov partitions and of SRB-measures.

Let us describe the tree last properties.

The geometric structure is given by regarding the set of the local stable and unstable manifolds: they depend continuously on the point. Moreover if the compact set K is an attractor ($K = \cap_{n \geq 0} f^n(V)$ for some neighborhood V), then we can cover K by finitely many open sets U whose intersection with K is homeomorphic to the product of $\mathbb{R}^d$ with a compact set T, such that $\mathbb{R}^d \times \{t\}$ corresponds to a local unstable manifold, for every $t \in T$. In other words K has a laminar structure whose leaves are unstable manifolds.

The existence of a Markov partition implies that a generic subset of K is conjugated to a shift on a space of words with finitely many letters. This implies the density of the set of periodic points, also it is a key point to construct the "thermodynamic" formalism. One can use this alphabet to show that K is the finite union of closures of orbits; the *transitivity classes* if the number of orbits is minimal.

An *SRB measure* of K is an invariant probability ν by the dynamical system, whose conditional measure to every local unstable manifold is absolutely continuous with respect to the Lebesgue measure. Using Birkhoff's theorem, we know that ν-almost every point of K has its orbit which intersects as much a Borelian U of K as $\nu(U)$. Whenever the dynamics is of class C^2, there exists actually a unique SRB measure on each transitivity class of K. Moreover the basin of the measure, that is the set of points which satisfy the previous recurrence property, is equal to a neighborhood of each transitivity component.

The last property shows that deterministic dynamical system may have (robust) statistical behaviors.

For a while, these (uniformly) hyperbolic dynamical system were conjectured to be generic: generically the non-wandering set of a dynamical system is hyperbolic (if moreover it is locally maximal, then the dynamical systems is said to be Axiom A). However this conjecture appears to be wrong. It is actually easy to see that conservative dynamical system cannot have a hyperbolic attractor without being Anosov. A counter example of this conjecture was found by Newhouse for (dissipative) C^2-dynamical systems [New74]. The idea is the following. We construct a horseshoe K of a surface such that its local stable and unstable sets are the product of a Cantor set T^s with $\mathbb{R}$ and the product of $\mathbb{R}$ with a Cantor set T^u respectively. We suppose that the sum of the Hausdorff dimension of T^u and T^s is everywhere greater than one: $d_s + d_u > 1$. Then we patch them to create a quadratic tangency between one local stable manifold with one local unstable manifold. Then the tangency appears to be robust: for an open set of C^2-perturbations there exists a tangency between a local stable manifold with a local unstable manifold (see [dAMY01]). The tangency points are

actually nonwandering and cannot be included in a hyperbolic compact set. Moreover for generic perturbations, we have the coexistence of infinitely many attracting or repelling points.

In the meantime, Hénon [Hén76] exhibits numerically a non-hyperbolic attractor for the family of maps $(x, y) \mapsto (x^2 + a + y, bx)$. Then Benedicks-Carleson [BC91] show its existence mathematically for a Lebesgue measure positive set of parameters (a, b). Viana-Mora show that this proof can be adapted to study many homoclinic tangencies [Mv93]. Later Benedicks-Young show the existence of an SRB for these parameters [BY93].

The work of [BC91] was perhaps the greatest progress in dynamical systems of the last decades, especially for the analysis developed. Unfortunately the definition of the dynamical systems satisfying these proofs are given by a long induction and so are difficult to be popularized and generalized. One has to understand the whole induction to state an extra property. In this work, we take advantage of a question of Pessin-Yurchenko on endomorphisms close to the Hénon's family useful for reaction-diffusion PDE of chemistry and physic. We prove in particular the existence of an SRB measure for this family and the Hénon's family. Contrarily to the previous generalization of [BC91], our approach is radically different, although we use many of the analytical tool of it (and some improvements of [YW01]). As a matter of fact, we only need two derivatives instead of three as for the previous proofs. In particular the proof is split into many independent parts. The proof is given for an open set of family of C^2-endomorphisms, that is maps not necessarily invertible and with possibly a wide critical set.

The presentation of the proof is split into three main parts that are basically independent.

In the first part of this work we state the definition of *regular dynamics*. This definition is completely intrinsic to the attractor. For the Hénon map, the attractor is the union of the unstable manifolds of a Horseshoe. When the parameter are regular, the attractor is spanned by a compact, continuous family of curves Σ endowed with an (infinite) Markov partition $\mathcal{Y}$ satisfying some analytic properties. Such a structure $(\Sigma, \mathcal{Y})$ is called a *regular puzzles algebra*. It is independently stated at the beginning of the first part. Then we show the existence of an SRB measure. It should be reasonable to obtain from this definition many other properties of the dynamics: exponential decay of correlation of the SRB, thermodynamical formalism...

In the second part, we define the *strongly regular dynamics*. This definition regards the embedding of a (not necessarily regular) puzzle algebra (A, Σ). It asks for the existence of quadratic tangency between some combinatorially defined local unstable manifolds of A with some combinatorially defined stable manifolds of A. The combinatorial hypothesis is very close to the proof of Jackobson's theorem by Yoccoz [Yoc]. The main difficulty of this part is to prove that strongly regular dynamics are regular. This provides actually a connection between the combinatorial properties of these algebras to their analytical properties. In particular, we realize that the Markov partition has actually a locally compact speudo-semi-group structure. We call this link the combinatorial rigidity. Eventually we prove that the strongly regular dynamics preserve a unique SRB measure which is physical and ergodic, following arguments close to [Bv06].

In the last part, we take advantage of the algebraic structure of strongly regular maps to construct a sequence families of local unstable manifolds $(\check{\Sigma}_k)_k$ whose cardinalities growth exponentially and

fill exponentially fast the set of the local unstable manifold of a fixed point. Then we compare the motion of these unstable manifolds with respect to the parameter a to the one of partial Markov partition (the puzzle algebra $(\Sigma_k, \mathcal{Y}_k)$). The evaluation of the set of the good parameters is given by a large deviation argument.

Part I

Regular dynamics

2 Basic definitions

When B is zero, the non-uniform hyperbolic behavior of $f_{a,0}$ for a large set of parameters a, is well understood as the celebrated Jakobson's theorem on the quadratic map $f_a :\ x \mapsto x^2 + a$. For $a \in (-2, -3/4)$, the map f_a has exactly two hyperbolic fixed points $\alpha^0 := \frac{1-\sqrt{1-4a}}{2}$ and $\beta^0 := \frac{1+\sqrt{1-4a}}{2}$. As these points are hyperbolic, for b small enough, $f_{a,B}$ has exactly two hyperbolic fixed points $\mathcal{A}$ and $\mathcal{B}$ close to $(\alpha^0, 0)$ and $(\beta^0, 0)$ respectively.

For the sake of simplicity, sometimes $f_{a,B}$ will be denoted by f. Also we denote by M the minimal integer such that $f_a^{M+1}(0)$ is in $[\alpha^0, -\alpha^0]$. We will show later that M is of the order of $-\log_4(2 + a)$. Since a is close to -2, M is a large integer. real number b is taken small after that M is taken large, such that $\theta := \frac{1}{|\log b|}$ is small with respect to 2^{-M}.

2.1 Rays

For $a \in (-2, -3/4)$, we saw that the quadratic map $f_a :\ x \mapsto x^2 + a$ has exactly two hyperbolic fixed points α^0 and β^0. The point $\alpha^1 := -\alpha^0$ is as well sent by f_a to α^0. The point α^1 is backward attracted by β^0. We denote by α^n the preimage of α^0 by $(f_{a|[0,\beta^0]})^n$, for $n \geq 1$. The sequence $(\alpha^n)_{n\geq0}$ is well defined and converges to β^0. We notice that the point $\pm\alpha^i$ is sent by f_a^i to α^0 but not by f_a^{i-1}.

We remark that the critical value a of the quadratic maps f_a belongs to a segment $[-\alpha^{M+1}, -\alpha^M]$.

Let us denote by $\pm\hat\alpha^n$ the preimages of $-\alpha^n$ in $[\alpha^0, \alpha^1]$, with $\hat\alpha^n > 0$.

The point $\pm\hat\alpha^i$ is sent by f_a^{i+1} to α^0 but not by f_a^i. We notice that $\hat\alpha^1$ is α^1.

These preimages of the fixed point are the foundations of the real 1-dimensional Yoccoz puzzle, and also of the real 2-dimensional puzzle that we will construct. The first interest of this construction is that the compact set:

$$K_0 := \cup_{n\geq1}\{-\alpha^n, \alpha^n\} \cup \{-\beta^0, \beta^0\}$$

is hyperbolic. In particular K_0 is sent into itself by f_a. Also the following local stable manifolds of $f_{a,0}$:

$$\mathcal{L}_\alpha^0 := \left\{ (x,y) \in \mathbb{R} \times \left(-\frac{1}{2}, \frac{3}{4} \right) : \ x^2 + a + y = \alpha \right\}, \ \alpha_0 \in K_0,$$

from the leaves of a normally expanded and has its closure sent into itself. By *normally expanded* we mean that there exists n such that for any $x \in \mathcal{L}$, any unit vectors u and v, with u tangent to the leaves and v normal to the leaves we have:

$$\| p \circ Tf_{a_0,0}^n(u) \| > 2(\| Tf_{a_0,0}^n(v) \| + 1),$$

with p the projection on the normal bundle of the leaves.

So for b small, this trivial lamination is persistent.[1] Here the *persistence* means that for every f

Figure 2: Dynamics between the rays illustrated by the arrows.
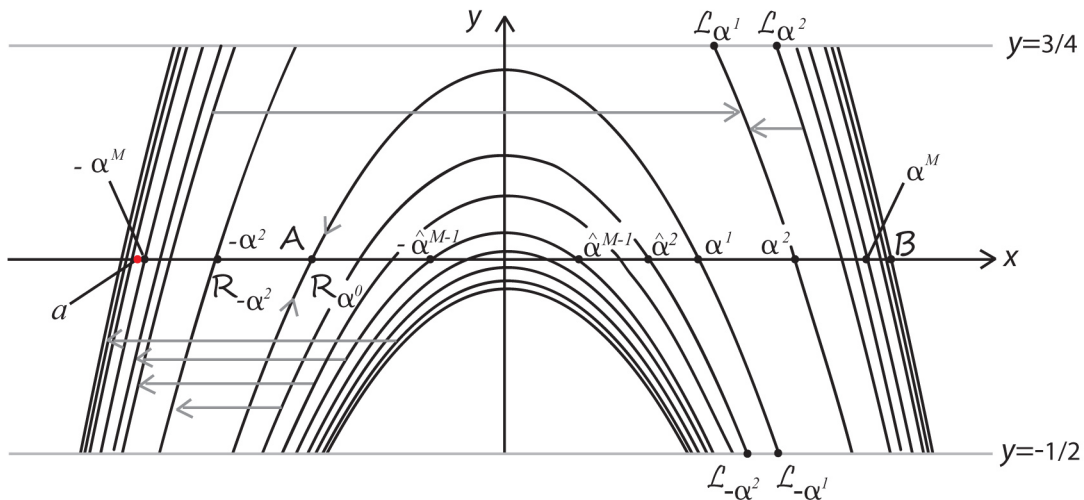
C^1-close to $f_{a,0}$, for each $\alpha \in K_0$, there exists a curve $\mathcal{L}_\alpha$ satisfying the following properties:

- $\mathcal{L}_\alpha$ is C^1-close to $\mathcal{L}_\alpha^0$,

- f sends the closure of $\mathcal{L}_\alpha$ into $\mathcal{L}_{f_a(\alpha)}$,

- the boundary of each $\mathcal{L}_\alpha$ is located on the lines $y = -\frac{1}{2}$ or $y = +\frac{3}{4}$.

We notice that the above notation is quite confusing since $\mathcal{L}_{\alpha^i}^0$ contains $(\alpha^{i+1}, 0)$, but not $(\alpha^i, 0)$. Also in order to simplify our study, we shall restrict the domain of our study. Let b small with respect to 2^{-2^M} s.t. the line $\{y = 2\theta\}$ intersects each curve $(\mathcal{L}_{\pm\alpha_i})_{i=0}^{M-1}$ at exactly two points, with $\theta = \frac{1}{|\log b|}$.

For $n \geq 0$, let $\mathcal{R}_{\alpha^{n+1}}$ and $\mathcal{R}_{-\alpha^{n+1}}$ be the connected components of $\mathcal{L}_{\alpha^n} \cap (\mathbb{R} \times [-2\theta, 2\theta])$, which are close to $(\alpha^{n+1}, 0)$ and $(-\alpha^{n+1}, 0)$ respectively.

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[1]See [Ber08] for a general theory on persistent laminations in the endomorphisms context.

For $n \in \{0, \cdots, M-1\}$, let $\mathcal{R}_{\hat{\alpha}^n}$ and $\mathcal{R}_{-\hat{\alpha}^n}$ be the connected components of $\mathcal{L}_{-\alpha^n}$ which is close to $(\hat{\alpha}^n, 0)$ and $(-\hat{\alpha}^n, 0)$ resp. Since α^0 is equal to $-\alpha^1$, the ray $\mathcal{R}_{\alpha^0}$ is equal to $\mathcal{R}_{-\alpha^1}$.

In analogy with the Yoccoz' Puzzle for complex quadratic maps, we call these local stable manifolds $(R_\alpha)_\alpha$ *rays*.

2.2 First Partition

A second interest of this construction is that it already gives us the flavor of the dynamics of $f = f_{a,B}$. We recall that f_a is real map $x \mapsto x^2 + a$. For the following description, the reader look at picture 3.

- Let Y_e be the set bounded by the rays $\mathcal{R}_{\alpha^0}$ and $\mathcal{R}_{\alpha^1}$ and the lines $\{y = -2\theta\}$ and $\{y = 2\theta\}$, with $\theta := \frac{1}{|\log b|}$.

- Let $Y_{w_+^n}$ be the set bounded by the rays $\mathcal{R}_{\alpha^n}$ and $\mathcal{R}_{\alpha^{n+1}}$ and bounded by the lines $\{y = \pm 2\theta\}$. The segment $[\alpha^n, \alpha^{n+1}]$ is sent univalently by f_a^n to $[\alpha^0, \alpha^1]$. So the set $Y_{w_+^n}$ has its image by f^n that stretches across Y_e.

- Similarly, the set $Y_{w_-^n}$ bounded by the rays $\mathcal{R}_{-\alpha^n}$ and $\mathcal{R}_{-\alpha^{n+1}}$ and by the lines $\{y = \pm 2\theta\}$ has its image by f^n that stretches across Y_e.

- Also, for $n \in \{2, M-1\}$, the set $Y_{s_+^n}$ (resp. $Y_{s_-^n}$) bounded by the lines $\{y = -2\theta\}$ and $\{y = 2\theta\}$ and bounded by the rays $\mathcal{R}_{\hat{\alpha}^{n-1}}$ and $\mathcal{R}_{\hat{\alpha}^n}$ (resp. by the rays $\mathcal{R}_{-\hat{\alpha}^{n-1}}$ and $\mathcal{R}_{-\hat{\alpha}^n}$), has its image by f^n that stretch across Y_e. The sets $Y + s_+^n$ and $Y_{s_-^n}$ have their images by f^n that stretch across Y_e.

We note that $Y_{s_\pm^n}$ is included in Y_e, whereas $Y_{w_\pm^n}$ is outside of Y_e.

We denote by $\mathcal{H}$ the union of some of these pieces:

$$\mathcal{H} := \bigcup_{n \geq 1} \left[Y_{w_+^n} \cup Y_{w_-^n} \right] \cup \bigcup_{n=2}^{M-1} \left[Y_{s_+^n} \cup Y_{s_-^n} \right]$$

Let $\mathcal{I}$ be the set bounded by $\mathcal{R}_{\hat{\alpha}^{M-1}} = \mathcal{R}_{-\hat{\alpha}^{M-1}}$ and the lines $\{y = \pm 2\theta\}$. Let $\mathcal{D}$ be the union of $\mathcal{H}$ with $\mathcal{I}$. We notice that $\mathcal{D}$ is bounded by a local stable manifold of $\mathcal{B}$ and the lines $\{y = \pm 2\theta\}$.

We notice that $\mathcal{I}$ is included in Y_e. Its image by f is included in $Y_{w_-^M} \cup Y_{w_-^{M-1}}$.

Actually the region $\mathcal{H}$ is 'nice', since it is somehow uniformly hyperbolic though not f-invariant: for a suitable metric on $\mathbb{R}^2$, we have $|p_x \circ \partial_x f(z)| > 1$ for every point $z \in \mathcal{H}$, with p_x the first coordinate projection of $(x, y) \in \mathbb{R}^2 \mapsto x$.

Let χ be the cone field on $\mathcal{D}$ such that $\chi(z)$ is of radius θ and centered at $\partial_x := (1, 0)$.

Let us now compute some estimates of the derivatives of f:

Proposition 2.1. *Let $\epsilon := \frac{1}{\sqrt{M}}$. Let $c = \log(2) - \epsilon$ and $c^+ := \log(2) + \epsilon$. For M sufficiently large and then b sufficiently small we have:*

1. *The norm $\|Tf_{|\mathcal{D}}\|$ of the differential of f is less than e^{2c^+}, the norm $\|T^2 f_{|\mathcal{D}}\|$ of the Hessian of f is less than e^{c^+},*
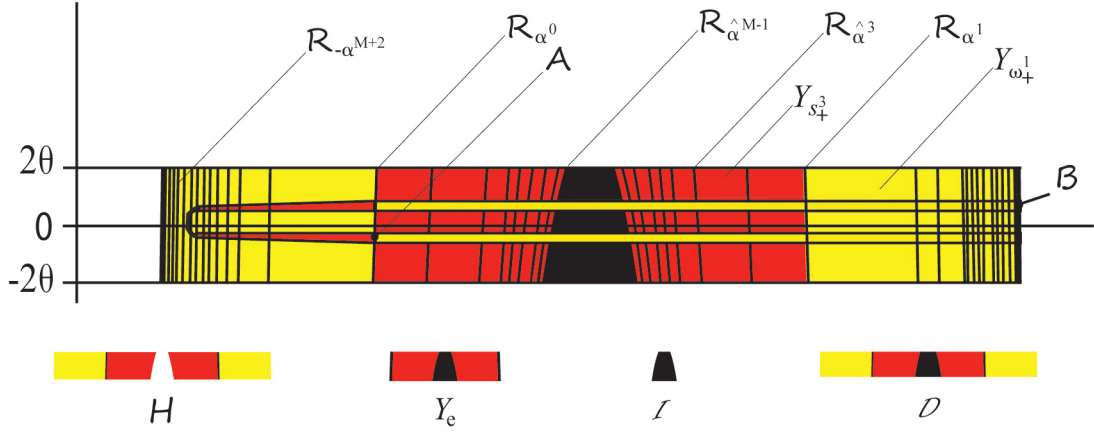
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Figure 3: Hyperbolic and critical strip and their images.

2. *for every point $z \in Y_{s_{\pm}^n}$, every unit vector $u \in \chi(z)$, $m \leq n$:*

$$\|T_z f^{n-m}(u)\| \geq e^{c(n-m)} \quad \text{and} \quad e^{c^+ n} \geq \|T_z f^n(u)\| \geq e^{cn},$$

3. *for every point $z \in Y_{w_{\pm}^M} \cup Y_{w_{\pm}^{M-1}}$, every unit vector $u \in \chi(z)$:*

$$e^{2cM} \leq \|T_z f^n(u)\| \leq e^{2c^+ M},$$

4. *on $\mathcal{D}$, the angle between e_1 and the horizontal line has a derivative with respect to $\partial_x = (1,0)$ close to -1, when b is small.*

Proof. The first statement follows from an easy computation.

Statement 4) follows from the fact that $e_1(x,y)$ is $(1,x)$, when $B = 0$.

2) As $Tf^m(u)$ belongs to χ for $1 \leq m \leq n$, it is easy to see that:

$$\|T_z f^{n-m}(u)\| > e^{c(n-m)}.$$

The case $m = 0$, and the remaining inequality (3) are proved in the appendix. $\square$

To prove the non-uniform hyperbolicity of f, we show that for almost every point z of the unstable manifold of $\mathcal{A}$, we show that every tangent vector u is expanded exponentially fast by $(T_x f^n)_n$. To this end we will show that for most times n, the vector $T_z f^n(u)$ is in χ and that $f^n(z)$ is in $\mathcal{H}$.

To control what happen after that the point z is sent by some iterate f^n into $\mathcal{I}$, we will construct other cone fields around some flat curves in Y_e.

2.3 Flat and stretched curves settings

We fix for the rest of the proof a C^2-identification of Y_e with $[-1,1] \times [-2\theta, 2\theta]$ such that the function of C^1-norm less than θ are those included in $Y_e \cap \mathrm{R} \times [-\theta, \theta]$ and with tangent space in χ.

A *flat curve* S is a C^1 curve whose tangent space is Lipschitz, such that:

- S is included in $Y_e \cap \mathbb{R} \times [-\theta, \theta]$,

- the tangent space of S belongs to χ,

- in the identification of Y_e, S is the graph of a function defined on an interval of $[-1,1]$ with derivative Lipschitz of constant less than θ.

A curve in Y_e is *stretched* if its stretches across Y_e. This means that its end points belong to the rays $\mathcal{R}_{\alpha^1}$ and $\mathcal{R}_{-\alpha^1}$.

We notice that the flat stretched curve are the curve equal to the graph of a function of norm less than θ and with θ-Lipschitz derivative.

For instance the intersection $\mathbb{S}$ of Y_e with the maximal local unstable manifold of $\mathcal{A}$ between the rays $R_{-\alpha^{M-1}}$ and $R_{\alpha^{M-1}}$ is a flat and stretched curve, for b-small enough.

2.4 Setting of some scales

We have already defined a few constants. We are going to introduce three other ones. Let us give a list of these constants, their meaning and how they are they are linked and fixed.

M, b, a, B   these constants refer to parameters for which the family of maps $(f_{a,B})_a$ is studied. *First M is fixed and large, then b is taken sufficiently small depending on M.* The parameter a is taken such that the critical point of the quadratic map f_a returns in $[\alpha^0, \alpha^1]$ in $M+1$ iterations. The function B has C^2-norm less than b and the Jacobian of f is less than b.

θ   is equal to $\frac{1}{|\log b|}$ and is related to all the geometric proportion: flat curves, height of Y_e,...

c, $c^{\pm}$ The constant c is equal to $\log 2 - \epsilon$, with $\epsilon := \frac{1}{\sqrt{M}}$. We recall that $\log 2$ is the Lyapunov exponent of f_{-2}. Also for $z \in Y_{s_{\pm}^n}$ and $u \in \chi(z)$, with $1 \leq n \leq M-1$,

$$e^{cn}\|u\| \leq \|T_z f^n(u)\| \leq e^{c^+ n}\|u\|$$

We will use also the following expressions:

$$c^+ := \log 2 + \epsilon, \quad c^- = \log 2 - 2\epsilon, \quad c^{++} := \log 2 + \sqrt{\epsilon}, \quad c^{--} := \log 2 - \sqrt{\epsilon}$$

μ   real number which depends on M via the following formula $\mu := e^{-\sqrt{M}}$.

K   is a numerical constant that cleans up our computations. It does not depend on M nor b.

$C(M)$ is a constant that depends only on M (and not on b). It is related to distortion properties that we will state later.

The order in which we chose the constants is the following: first M is chosen large (or equivalently $2 + a$, μ, are chosen small), and then b is is chosen small.

3 Regular dynamics

3.1 Puzzle pieces

The puzzle pieces are always associated to a flat and stretched curve S.

Definition 3.1. A *puzzle piece* α of S is the data of:

- an integer n_α called the *order* of α,

- a segment S_α of S sent by f^{n_α} to a flat stretched curve S^α.

Example 3.2 (Simple pieces)**.** For b small enough, for any flat stretched curve S, each pair $s_\pm^n := \{Y_{s_\pm^n} \cap S, n\}$, for $2 \le n \le M - 1$ or $e := \{S, 0\}$ define a puzzle piece. The pieces $\{s_\pm^n\}_{n=2}^{M-1}$ are called *simple*. We put $S_{s_\pm^n} := Y_{s_\pm^n} \cap S$ and $S^{s_\pm^n} := f^n(S_{s_\pm^n})$.

Proof. Let s be a simple piece and let S be a flat stretched curve. By the local invariance of χ we know that S^s has its tangent space in χ. Also by definition of the rays that bound Y_s it is clear that S^α is stretched. The only difficulty in this example is to prove that S^s is a flat for every simple pieces s. This flatness follows from Property (2) of Proposition 2.1 and the following lemma:

Lemma 3.3 ([YW01], lemma 2.4, where in [BC91]???)**.** *Let $\gamma_0\ [0,1] \to \mathbb{R}^2$ be a C^2 curves and let $\gamma_i(s) = f^i \circ \gamma$. We denote by $k_i(s)$ the curvature of γ_i at $\gamma_i(s)$. Let $\kappa > b^{1/3}$. If for every s, $k_0(s) \le 1$ and*

$$\|Tf^j(\gamma_{n-j}(s))\gamma'_{n-j}(s)\| \ge \kappa^j \|\gamma'_{n-j}(s)\|, \quad \forall j \le n.$$

Then $k_n(s) \le Kb/\kappa^3$.

$\square$

Definition 3.4. Let S be a flat and stretched curve. A puzzle piece α of S is *regular* if the two following properties hold:

h-time For every $z \in S_\alpha$ and $w \in T_z S_\alpha$ and every $l \le n_\alpha$, the following *hyperbolic times inequality* is satisfied:

$$\|T_{z_l} f^{n_\alpha}(w_l)\| \ge e^{\frac{c}{3}(n_\alpha - l)} \cdot \|w_l\|,$$

with $z_l := f^l(z)$ and $w_l := Tf^l(w)$.

Dist For all $z, z' \in S_\alpha$, all unit vectors $w \in T_z S$ and $w' \in T_z S'$, we have:

$$\left| \log \frac{\|T_z f^{n_\alpha} w\|}{\|T_{z'} f^{n_\alpha} w'\|} \right| \leq C(M) d\big(f^{n_\alpha}(z), f^{n_\alpha}(z')\big) \sum_{i=0}^{n_\alpha - 1} e^{-i \cdot \frac{c}{4}}$$

Example 3.5 (Regularity of simple pieces). Each simple piece $s \in \{s_\pm^n\}_{n=2}^{M-1}$ is regular.

Proof. The h-time property follows from Proposition 2.1. The *Dist*-property is easy to be shown since the constant C depends on M, and the order of the simple pieces is less than M. $\qquad\square$

We fix definitively the constant $C = C(M)$ after M but before b, such that for every simple pieces of any flat stretched curves is regular.

Remark 3.6. Using the Schwarzian derivatives properties of the quadratic maps, one can find C independent of M, for C^3 perturbations B. We prefer to work with B of only of class C^2.

Operation $\star$ on puzzle pieces Let $\alpha := \{S_\alpha, n_\alpha\}$ and $\beta = \{S_\beta^\alpha, n_\beta\}$ be two puzzle pieces of respectively S and $S^\alpha := f^{n_\alpha}(S_\alpha)$. We define the puzzle piece of S: $\alpha \star \beta := \{(f_{|S_\alpha}^{n_\alpha})^{-1}(S_\beta^\alpha), n_\alpha + n_\beta\}$.

Let us notice furthermore that $\alpha \star \beta$ is regular if α and β are regular. The h-time property of $\alpha \star \beta$ is obvious. The Dist-property follows from the following computation. Let w, w' be two unit vectors at respectively $z, z' \in S_{\alpha \star \beta}$:

$$D := \left| \log \frac{\|T_z f^{n_{\alpha \star \beta}}(w)\|}{\|T_{z'} f^{n_{\alpha \star \beta}}(w')\|} \right| \leq \left| \log \frac{\|T_z f^{n_\alpha}(w)\|}{\|T_{z'} f^{n_\alpha}(w')\|} \right| + \left| \log \frac{\|T_{f^{n_\alpha}(z)} f^{n_\beta}(w_1)\|}{\|T_{f^{n_\alpha}(z')} f^{n_\beta}(w_1')\|} \right|,$$

with w_1 and w_1' the unit vector in the direction of $T_z f^{n_\alpha}(w)$ and $T_{z'} f^{n_\alpha}(w')$ respectively. By using the Dist-property of the puzzle pieces α, β and the h-time property of α we get:

$$D \leq Cd(f^{n_{\alpha \star \beta}}(z), f^{n_{\alpha \star \beta}}(z')) \sum_{i=0}^{n_\beta - 1} e^{-ic/4} + Cd(f^{n_\alpha}(z), f^{n_\alpha}(z')) \sum_{i=0}^{n_\alpha - 1} e^{-ic/4}$$

$$\leq Cd(f^{n_{\alpha \star \beta}}(z), f^{n_{\alpha \star \beta}}(z')) \sum_{i=0}^{n_\alpha + n_\beta - 1} e^{-ic/4}$$

Puzzle Algebra A *puzzle algebra* is the data of a family $\Sigma = (S^t)_{t \in T}$ of flat stretched curves S endowed with regular puzzle pieces $\mathcal{Y}(S)$ such that:

- The space T is endowed with a compact metric topology such that $t \mapsto S^t$ is continuous for the C^1-topology,

- for all $S \in \Sigma$, $\alpha \in \mathcal{Y}(S)$, the curve S^α belongs to Σ. Moreover the following map is bijective:

$$(S, \alpha) \in \prod_{S \in \Sigma} \mathcal{Y}(S) \mapsto S^\alpha \in \Sigma.$$

- for all $S \in \Sigma$, $\alpha, \beta \in \mathcal{Y}(S)$, the segments S_α and S_β have disjoint interiors in S.

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We denote by $\mathcal{Y} := \cup_{S \in \Sigma} \mathcal{Y}(S)$.

Example 3.7 (Simple puzzle algebra). Let T_1 be the set of the presequences $(s_i)_{i \leq 0}$ of simple pieces. We notice that for every $t = (s_i)_{i \leq 0} \in T_1$, for any flat stretched curve S, the sequence $(S^{s_{-n} \star \cdots \star s_0})_{n \geq 0}$ converges in the C^1-topology to a flat stretched curve S^t that does not depend on S.

For $t = (s_i)_{i \leq 0}$, $t' = (s'_i)_{i \leq 0}$ let $t \wedge t'$ be the maximal integer $d \geq 0$ s.t. $s_{-i} = s'_{-i}$ for all $i < d$. The space T_1 endowed with the metric $t, t' \mapsto b^{t \wedge t'/8}$ is compact. Moreover the family $\Sigma_1 := (S^t)_{t \in T_1}$ is a continuous family of flat stretched curves (we will show that this family is actually 1-Lipschitz).

We endow each curve S of Σ_1 with the set $\mathcal{Y}_1(S)$ of the simple pieces. We identify for $s \in \mathcal{Y}(S^t)$, the curve $(S^t)^s$ to $S^{t'}$ where t' is the concatenation of t with s.

We notice that $(\Sigma_1, \mathcal{Y}_1)$ is a puzzle algebra. We call it the *simple puzzle algebra*.

3.2 Definition of regular dynamics

An f-stable set A is *regular* if there exists a puzzle algebra $(\Sigma, \mathcal{Y})$ such that:

1. A is equal to the union $\cup_{n \geq 0, \ S \in \Sigma} f^n(S)$.

2. The subset $\cup_{\alpha \in \mathcal{Y}(S)} S_\alpha$ is of full measure in S, moreover the Lebesgue measure of the complement of $\cup_{\alpha : n_\alpha \leq n} S_\alpha$ is less than $e^{-nc/4}$ for every $n \geq M$, for every $S \in \Sigma$.

3. For every $S^t \in \Sigma$ and $n \geq 0$, there exists a neighborhood V_t of $t \in T$, and a continuous family of embeddings $(\phi_{t'})_{t' \in V_t}$ from S^t onto $S^{t'}$ s.t. $\phi_{t'}$ sends the puzzle pieces $\{S^t_\alpha : n_\alpha \leq n, \alpha \in \mathcal{Y}(S^t)\}$ onto the puzzle pieces $\{S^{t'}_{\alpha'} : n_{\alpha'} \leq n, \alpha' \in \mathcal{Y}(S^{t'})\}$ for $t' \in V_t$. Moreover $(S^{t'\alpha'})_{t' \in V_t}$ corresponds to an open subset of T.

The dynamics is *regular* if there exists a regular set A s.t. $\Sigma, \mathcal{Y})$ contains $(\Sigma_1, \mathcal{Y}_1)$. This implies that A contains all the unstable manifold of the hyperbolic set $\cap_{n \geq 0} f^n(\mathcal{H})$ (which is basically a horseshoe).

Remarks 3.8. Let us notice that:

- A may be non compact, however we call it attractor.

- $\tilde{X} := \prod_{t \in T} S^t \times \{t\}$ is a lamination which is canonically immersed in Y_e via the first coordinate projection. Let X be the image of this lamination.

- Each curve of Σ is a local unstable manifolds of f.

- The following return map is well defined and measurable:

$$R : \tilde{X} \to \tilde{X}$$

$$x \mapsto f^{n(x)}(x),$$

where $n(x) > 0$ is the order of the puzzle piece of $\mathcal{Y}(S)$ containing $x \in S$ in its interior, or 0 if there is not such a piece.

Example 3.9 (Tchebychev map). The map $f_{-2,0} : (x, y) \mapsto (x^2 + y - 2, 0)$.

We compactify the set of the simple pieces $\{s^i_\pm\}_{i\geq 2}$ by adding the point s^∞. We define on the complement of the diagonal of $\{s^i_\pm\}_{i\geq 2} \cup \{s^\infty\}$ the distance:

$$d : (s^i_\delta, s^{i'}_{\delta'}) \in \left(\{s^i_\pm\}_{i\geq 2} \cup \{s^\infty\}\right)^{-\mathbb{N}} := b^{min(i,i')}.$$

Let T be the set of the presequences $(a_i)_i \in \left(\{s^i_\pm\}_{i\geq 2} \cup \{s^\infty\}\right)^{-\mathbb{N}}$ such that if $a_i = s^\infty$ then $a_j = s^\infty$ for every $j < i$. We endow T with the distance

$$d((a_i)_i, (a'_i)_i = d(a_j, a'_j)b^j,$$

with $j \leq 0$ maximal such that $a_i = a'_i$ for every $i > j$. We notice that T is compact.

We notice that $f_{-2,0}$ is regular with:

- $A := [-2, 2] \times \{0\}$,

- $\Sigma := (S^t)_{t \in T}$ with $S^t := [-1, 1] \times \{0\}$,

- $\mathcal{Y}(S)$ formed by the simple pieces s^n_+ or s^n_- with $n \geq 2$, for every $S \in \Sigma$

For every $t \in T$ and every $\alpha \in \mathcal{Y}(S^t)$, we put $(S^t)^\alpha = S^{t'}$, with t' the concatenation of $t \cdot \alpha$.

This is the *only* example that we consider for which M is infinite.

3.3 Geometry of regular set

Let A be a regular set endowed with a puzzle algebra $(\Sigma, \mathcal{Y})$. Also for $t \in T$ and $\alpha \in \mathcal{Y}(S^t)$, by regular property (3), there exists an open neighborhood V_t of $t \in T$ and a family of puzzle pieces $(\alpha^{t'})_{t' \in V_t} \in \prod_{t \in V_t} \mathcal{Y}(S^{t'})$ such that $t' \in V_t \mapsto S^{t'}_\alpha$ is continuous. Moreover $O_\alpha := \prod_{t' \in V_t} int(S^{t'}_\alpha) \times \{t'\}$ is an open subset of $\tilde{X}$ mapped by R homeomorphicaly onto an open subset $\prod_{t' \in V_t} int(S^{t'\alpha}) \times \{t' \cdot \alpha\}$ of $\tilde{X}$, by (3).

The following space corresponds to the whole attractor, but unfolded:

$$\hat{A} := \mathbb{Z} \times \tilde{X} / \sim$$

with $\sim$ the equivalent relation spanned by $(n_\alpha + k, x, t) \sim (k, f^{n_\alpha}(x), t \cdot \alpha)$, if $x \in S^t_\alpha$ with $\alpha \in \mathcal{Y}(S^t)$, for every $k \in \mathbb{Z}$.

We remark that the equivalent classes are closed and so $\hat{A}$ is a Hausdorf space. The leaves of $\mathbb{Z} \times \tilde{X}$ are glued together in the following rule. If (S^t_α, n) is patched to $(S^t_{\alpha'}, n')$ then the set correspond to one of these curves and a segment of the other. Consequently $\hat{A}$ support a canonical partition by 1-dimensional leaves. However, in general a transversal structure to these leaves do not exists, and so the leaves do not form a lamination on $\hat{A}$.

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There is a natural map $p : (n, x, t) \in \hat{A} \mapsto f^n(x)$. Let $\hat{f} := (n, x) \in \hat{A} \mapsto (n + 1, x)$. We notice that the following diagram commutes:

$$
\begin{array}{ccc}
 & \hat{f} & \\
\hat{A} & \rightarrow & \hat{A} \\
p \downarrow & & \downarrow p \\
\mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\
 & f &
\end{array}
$$

3.4   Existence of an SRB measure

Definition of SRB measure   The definition of SRB measure in uniformly hyperbolic dynamics is well established. A uniformly hyperbolic attractor A has a *lamination structure*: it is covered by open subsets $(U_i)_i$ endowed with homeomorphisms $\phi_i : U_i \to \mathbb{R}^d \times T_i$ such that for every $t \in T_i$, the subset $\mathbb{R}^d \times \{t\}$ corresponds to a local unstable manifold. An *SRB* measure is an f-invariant measure absolutely continuous with respect to the leaves of this lamination.

Let us define the *absolutely continuous measure with respect to the leaves of a lamination*.

This definition is local and so shall be regarded in a trivialization $U_i \approx \mathbb{R}^d \times T_i$. Let $\mathcal{M}_i$ be the σ-algebra on $\mathbb{R}^d \times T_i$ generated by the products $A \times B$ of Borelian sets $A \subset \mathbb{R}^d$ and $B \subset T_i$.

For a measure ν_i on U_i, we define $\hat{\nu}_i := \pi_2^* \nu_i$ the measure on T_i given by pushing forward ν_i by the projection $\pi_2 : U_i \to T_i$.

Let $m \times \hat{\nu}_i$ be the measure defined on $\mathcal{M}_i$ by:

$$
(m \times \hat{\nu}_i)(A, B) = m(A) \cdot \hat{\nu}_i(B),
$$

where m is the Lebesgue measure of $\mathbb{R}^d$.

A measure ν is *absolutely continuous with respect to the leaves of a lamination* if for every i there exists an $\mathcal{M}_i$-measurable function ρ_i on $R^d \times T_i$ such that $\nu|U_i = \rho_i \circ \phi_i \cdot m \times \hat{\nu}_i$, with $\nu = \nu|U_i$.

Then the conditional probability measures $\{\nu_t : t \in T_i\}$ of $\nu_{|U_i}$ relative to the transversal space T_i of U_i (see [Roh52]) are absolutely continuous with respect to m: one may take $\nu_t = (\rho_i|S^t)m$ with $S^t := \phi_i^{-1}(\mathbb{R}^d \times \{t\})$ for every $t \in T_i$.

The above definition can be generalized to regular attractors. As $\tilde{X} \times \tilde{X} \times \{0\}$ is canonically embedded in $\hat{A}$ and spans $\hat{A}$ by the action of $\hat{f}$. We say that a $\hat{f}$-invariant measure on $\hat{A}$ is absolutely continuous with respect to the leaves of $\hat{A}$ if its restriction to $\tilde{X}$ is absolutely continuous with respect to the leaves of $\tilde{X}$.

We remark that an invariant measure for $\hat{f}$ on $\hat{A}$ pushes forward by p to an f-invariant measure supported by A.

Also we note that that R is the first return map at $\tilde{X} \times \{0\}$ for $\hat{f}$. Thus we start by constructing first an 'SRB measure' for R on $\tilde{X}$.

Let us recall some useful properties of R.

The following proposition is a mere consequence of the regular properties of the puzzle algebra:

Proposition 3.10. *The map R is uniformly expanding and has bounded distortion along unstable leaves:*

(1) $|\partial_1(R^k)'(x, S)|^{-1} \leq e^{-k \cdot \frac{c}{3}}$, *for every $k \geq 1$, every $S \in \Sigma$ and a.e. $x \in S$,*

(2) $|\partial_1(R^k)'(y, S)|/|\partial_1(R^k)(x, S)| \leq K$ *for any $k \geq 1$ and $S \in \Sigma$ and $x, y \in S$ whose k first R-iterates belong to the same piece of $\mathcal{Y}$.*

Measures absolutely continuous along unstable leaves for R   Let us construct a chart spanning the trivial lamination structure on $\tilde{X}$.

Let $\pi_2 : \tilde{X} \to T$ be the second coordinate projection.

We recall that Let $\pi_1 : \tilde{X} \to [-1, 1]$ the map whose restriction to every $S \in \Sigma$ is the first coordinate projection $[-1, 1] \times [-2\theta, 2\theta] \to [-1, 1]$, for the identification of Y_e with $[-1, 1] \times [-2\theta, 2\theta]$.

We identify $\tilde{X}$ to the product $[-1, 1] \times T$ via the map $(\pi_1, \pi_2) : \tilde{X} \to [-1, 1] \times T$.

Let $\mathcal{M}$ be the σ-algebra on $[-1, 1] \times T$ generated by the product $A \times B$ of measurable sets $A \subset [-1, 1]$ and $B \subset T$.

Given a Borel measure ν on $\tilde{X}$, let again $m \times \hat{\nu}$ be the measure on $\mathcal{M}$ defined by:

$$(m \times \hat{\nu})(A, B) = m(A) \times \hat{\nu}(B),$$

where m is the Lebesgue measure and $\hat{\nu} = (\pi_2)^* \nu$.

Proposition 3.11. *A regular dynamics f admits an SRB measure: there exists a measure μ on A, $\hat{f}$-invariant such that its restriction to $\tilde{X}$ is absolutely continuous with respect to the leaves of $\tilde{X}$.*

Remark 3.12. Actually $p^* \mu$ is an SRB measure of f.

Proof. By using Property 2 of regular dynamics, we only need to show the existence of an R-invariant measure μ on $\tilde{X}$ of the form $\mu = \rho(m \times \hat{\mu})$ which is bounded. We are going to proceed as in [Bv06].

For every $t \in T$, let m_t be the product $m \times \delta_t$, with δ_t the Dirac measure of T at t. Let us remind the following lemma:

Lemma 3.13 (4.9 in [Bv06]). *There exists $K > 0$ such that given any $t \in T$, the sequence $\lambda_n = R_*^n m_t$ satisfies:*
$$\lambda_n(A \times B) \leq K m(A) \hat{\lambda}_n(B),$$
for every $n \geq 1$, $A \times B \in \mathcal{A}$.

Since the measurable sets $A \times B$ generate the σ-algebra $\mathcal{M}$, the measure λ_n is absolutely continuous along the leaves, with Radon Nikodyn density ρ_n bounded by K_0. Moreover the same is true for the sequence:

$$\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j = \frac{1}{n} \sum_{j=0}^{n-1} R_*^j m_t.$$

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Since $\mu_n(A \times B) = \sum_j \lambda_j(A \times B) \le \frac{Km(A)}{n} \sum_j \hat{\lambda}_j(B) = Km(A)\hat{\mu}_n(B)$.

As $R^*\mu_n - \mu_n = \frac{1}{n}(\lambda_n - \lambda_0)$, any accumulation point μ of μ_n is a R-invariant measure. Such an accumulation point exists by compactness of the probability measures. The absolute continuity of μ is shown in Corollary 4.10 of [Bv06]. □

Part II

Strongly regular dynamics

Regular dynamics should be regarded as an intrinsic dynamical object. We do not know how often folds its laminar structure. We do not know how large is the basin of its SRB measure and nothing about its uniqueness[2]. Furthermore we do not have any information nor control on the perturbation of the dynamics. In particular, we do not know how far a segment of Σ or a puzzle piece of $\mathcal{Y}$ is going to persist and to be regular for small perturbations of the dynamics. Most of these problems do not exist in the one dimensional case in the C^3-topology.

For these reasons, we are going to define the strongly regular dynamics which will provide these extrinsic information on the attractor.

4 Extension of puzzle pieces

4.1 Definition

Let $\alpha = \{S_\alpha, n_\alpha\}$ be a regular puzzle piece of a flat stretched curves. We want to extend the regular properties of α to the curves that are close to S_α. Unfortunately as we deal in the C^2-topology we are unable to extend the *Dist*-property uniformly far with respect to n_α.

An *extension* of α is the data of:

- an embedding Y_α of the square $[0,1]^2$ into Y_e identified with its image and such that $[0,1] \times \{1/2\}$ corresponds to S_α,

- a cone field χ_α on Y_α,

such that the following conditions are satisfied:

Markov Y_α is sent by f^{n_α} into Y_e. We denote by $\partial^s Y_\alpha$ and $\partial^u Y_\alpha$ the subsets of the boundary of Y_α that correspond to $\partial[0,1] \times [0,1]$ and $[0,1] \times \partial[0,1]$ respectively. Then the two curves of $\partial^s Y_\alpha$ are sent by f^{n_α} into $\mathcal{R}_{\alpha^1}$ and $\mathcal{R}_{\alpha^0}$ respectively. The distance between $\partial^u Y_\alpha$ and S is greater than $\epsilon_\alpha := \theta^{n_\alpha}$. (see fig. 4.1)

---

[2]Actually we can prove that C^3-regular dynamics are physical: their basins are of Lebesgue positive measure, but a third derivative is useless when we consider strongly regular dynamics.

Cone For every point $z \in Y_\alpha$, for every point $z' \in S_\alpha$ at a minimal distance to z, the cone $\chi_\alpha(z)$ contains the translation of the cone centered at $T_{z'}S$ and with angle ϵ_α. Moreover for every $y \in [0,1]$, the curve that corresponds to $[0,1] \times \{y\}$ is of class C^1 with tangent space in χ_α. The map Tf^{n_α} sends χ_α into the cone centered at $(1,0)$ with angle 2θ (which contains χ).

h-times For every $z \in Y_\alpha$, every $w \in \chi_\alpha(z)$ and every $l \le n_\alpha$, the following hyperbolic times inequality is satisfied:

$$\|T_{z_l}f^{n_\alpha}(w_l)\| \ge e^{\frac{c}{4}(n_\alpha - l)} \cdot \|w_l\|,$$

with $z_l := f^l(z)$ and $w_l := Tf^l(w)$.

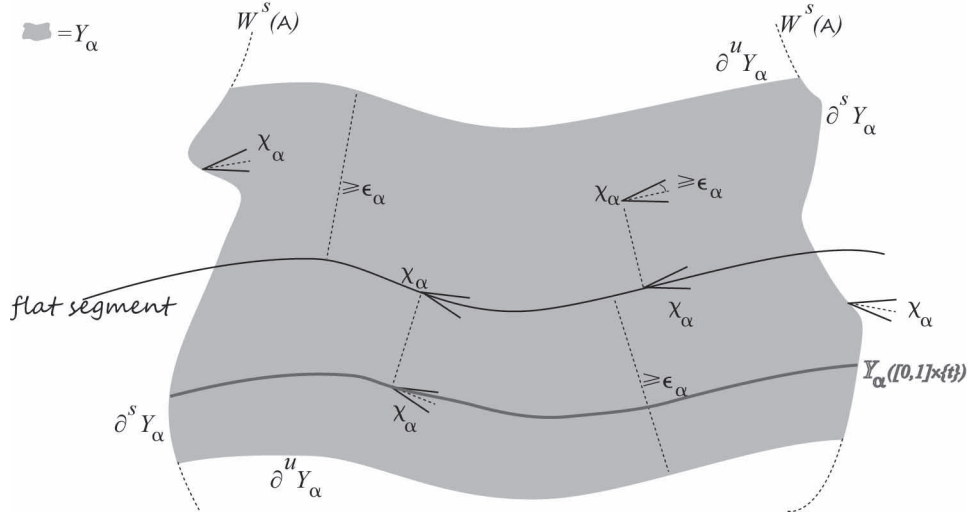**Remark 4.1.** By cone property, the tangent space of $\partial^u Y_\alpha$ is included in the complement of χ_α.

Figure 4: A regular extension of a puzzle piece.

4.2 Basic examples of extensions

Simple extensions Whenever α is simple, we endow it with the extension $\{Y_\alpha, \chi_\alpha\}$ where Y_α was defined in subsection 2.2 and χ_α the restriction to Y_α of the cone field χ centered at the horizontal line and with radius θ^{n_α}. We notice that all the regular properties of this extension are satisfied.

Canonical extensions Let $\alpha = \{S_\alpha, n_\alpha\}$ be a puzzle piece such that S_α is included in $\mathcal{I}$ (and so α is not simple). Let us describe the canonical extension of α. We recall that $\epsilon_\alpha := \theta^{n_\alpha}$.

Let V_α be ϵ_α neighborhood of S_α and $\hat{\chi}_\alpha$ be the cone field on V_α whose cone at $z \in V_\alpha$ is the union of the ϵ_α-neighborhood of $T_{z'}S_\alpha$ for every $z' \in S_\alpha$ which is ϵ_α-distant to z.

We notice that $z_i := f^i(z)$ and $z_i' := f^i(z')$ are at least $\epsilon_\alpha e^{2c+i}$ close for every $i \in \{0, \dots, n_\alpha\}$. thus we have

Claim 4.2. $\|T_z f^{n_\alpha} - T_{z'} f^{n_\alpha}\| \le n_\alpha \epsilon_\alpha e^{2c+n_\alpha}$

Proof.

$$\|T_z f^{n_\alpha} - T_{z'} f^{n_\alpha}\| \le \sum_{i=0}^{n_\alpha - 1} \|T_{z_i} f^{n_\alpha - i} \circ T_{z'} f^i - T_{z_{i+1}} f^{n_\alpha - i - 1} \circ T_{z'} f^{i+1}\|$$

$$\le \sum_{i=0}^{n_\alpha - 1} \|T_{z_{i+1}} f^{n_\alpha - i - 1}\| \cdot \|T_{z_i} f - T_{z'_i} f\| \cdot \|T_{z'} f^i\| \le \sum_i e^{2(n_\alpha - i - 1)c^+} \theta^{n_\alpha} e^{2c^+} e^{2ic^+} \le n_\alpha \epsilon_\alpha e^{2n_\alpha c^+}.$$

$\square$

This implies that every vector $u \in \hat{\chi}_\alpha$ cannot be sent to $T\mathcal{R}_{\alpha^1} \cup T\mathcal{R}_{\alpha^0}$. Thus both local stable manifolds in V_α of the ends points of S_α have their tangent space in χ_α^c. This implies that two maximal stable manifolds of these two points in V_α bounds with the translation by $(0, \epsilon_\alpha)$ and $(0, -\epsilon_\alpha)$ of S a domain Y_α homeomorphic to a square. Let us parametrize Y_α such that $[0,1] \times \{t\}$ corresponds to the segment of the translation of S by $2\epsilon_\alpha (t - \frac{1}{2})(0,1)$. Let χ_α be the restriction of $\hat{\chi}_\alpha$ to Y_α. By the above claim the h-time property is satisfied. Moreover the extension $\{Y_\alpha, \chi_\alpha\}$ is regular.

4.3   Operation $\star$ on extensions

Let Y_α and Y_β be two embeddings of the square $[0,1]^2$ into Y_e that we identify to their images.

Let us remark that $f_{|Y_\alpha}^{-n_\alpha}(Y_\beta)$ is canonically homeomorphic to the square if the following condition holds: For every $y \in [0,1]$ the curve γ_y in Y_α corresponding to $[0,1] \times \{y\}$ is immersed by f^{n_α} and its image intersects each component of $\partial^s Y_\beta$ at exactly one point; the segment of $f^{n_\alpha}(\gamma_y)$ between these two points is included in $Y_\beta \setminus \partial^u Y_\beta$. Then the canonical homeomorphism is given by an affine map from $[0,1]$ to a segment of $[0,1]$ composed by γ_y. We denote by $Y_\alpha \star Y_\beta$ this homeomorphism.

Then, we say that $\{Y_\alpha, n_\alpha\} \star \{Y_\beta, n_\beta\}$ is *well defined* and equal to the pair $\{Y_\alpha \star Y_\beta, n_\alpha + n_\beta\}$.

We remark that if the both curves of $\partial^s Y_\beta$ are sent by some iterate f^{n_β} into R_{α^0} and R_{α^1} respectively, then the two curves forming $\partial^s (Y_\alpha \star Y_\beta)$ are sent by $f^{n_\alpha + n_\beta}$ into R_{α^0} and R_{α^1} respectively. We remark also that for any closed subset S, the distance between $\partial^u (Y_\alpha \star Y_\beta)$ and S is at least the one between $\partial^u Y_\alpha$ and S.

Let us extend the operation $\star$ on puzzle pieces to their extension.

Proposition 4.3. *Let α be a regular puzzle piece of a flat stretched curve S, and β a regular puzzle piece of S^α. Let (Y_α, χ_α) and (Y_β, χ_β) and be puzzle extensions of α and β such that $\{Y_\alpha, n_\alpha\} \star \{Y_\beta, n_\beta\}$ is well defined and $\chi_{\alpha|Y_\alpha \star Y_\beta}$ is sent into χ_β by Tf^{n_α}.*

Then $\{Y_\alpha \star Y_\beta, \chi_{\alpha|Y_\alpha \star Y_\beta}\}$ is a regular extension of $\alpha \star \beta$. We denote it by $\{Y_{(\alpha \star \beta)}, \chi_{(\alpha \star \beta)}\}$.

We use brackets in the notation to emphasis on the fact that this extension is in non canonic nor simple, if $\beta \ne e$.

4.4   Geometry of extensions

By the Markov property, the geometry of the $\partial^s Y_\alpha$ for an extension $\{Y_\alpha, \chi_\alpha\}$ is given by the one of the stable manifolds. Contrarily to the uniformly hyperbolic case, this one can be very complicated. This is not the case with the following property:

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A point $z \in \mathbb{R}^2$ satisfies the (finite) *Collet-Eckmann condition* ($\mathcal{CE}^m$) if the following inequality holds:

$(\mathcal{CE}^m)$
$$\|Tf^j(z)(0,1)\| \geq e^{c^-j} \quad \forall j \leq m.$$

A point $z \in \mathbb{R}^2$ satisfies the (finite) *projective Collet-Eckmann condition* ($\mathcal{PCE}^m$) if the following inequality holds:

$(\mathcal{PCE}^m)$
$$\|Tf^j(z)(0,1)\| \geq e^{-jMc^+} \quad \forall j \leq m.$$

Let us define the equivalent finite concept for the stable manifold.

For $m \geq 1$, we denote by $\mathcal{F}_m$ the foliation integrating the most contractive direction of f^m. In other words, for $z \in \mathbb{R}^2$, its tangent space $T_z\mathcal{F}_m$ is the most contracted direction of $T_z f^m$ (*i.e.* the vector space associated to the smallest eigenvalue of $(T_z f^m)^* \circ T_z f^m$, where $(T_z f^m)^*$ is the adjoint of $T_z f^m$). This definition makes sense as soon as the linear function $T_z f^m$ is not the composition of a homothety with an isometry. We denote by e_m the unit vector field tangent to the leaves whose image by the projection on the first coordinate is $(1, 0)$, whenever it is possible.

We notice that $Tf(x,y) = \begin{bmatrix} 2x & 2 \\ 0 & 0 \end{bmatrix} + TB(x,y)$. Also $e_1 \sim (1, -x)$ and $\partial_x e_1 \sim (0, -1)$. The foliation $\mathcal{F}_m$ is said to be of order m.

From Collet-Eckmann to the geometry of $\mathcal{F}$ We can compare $\mathcal{F}_m$ and $\mathcal{F}_1$ at each point satisfying Condition $\mathcal{CE}^m$, as pointed out by the following proposition:

Proposition 4.4. *For b small enough and $e^{-\kappa} >> \sqrt{b}$, for every point $z \in \mathcal{D}$ and $N \geq 0$ that satisfies:*
$$\|T_z f^N\| \geq e^{-\kappa N},$$
we have whenever e_N, e_{N-1} are well defined:

1. $|\angle(e_N(z), e_{N-1}(z))| \leq (Ke^{-2\kappa}b)^{N-1}$,

2. $\|T_z f^i(e_N(z))\| \leq (Ke^{-2\kappa}b)^i$, *for $1 \leq i \leq N$,*

3. $\|d_z\angle(e_N(z), e_{N-1}(z))\| \leq (Ke^{-3\kappa}b)^{N-1}$.

Proof. See Lemma 2.1, Corollaries 2.1 and 2.2 of [YW01]. $\qquad\qquad\square$

Corollary 4.5. *Let $z \in \mathcal{D}$ satisfying $\mathcal{CE}^m$ (resp. $\mathcal{PCE}^m$), the vectors $(e_i(z))_{i=1}^m$ are well defined and we have:*

1. $|\angle(e_m(z), e_1(z))| \leq Kb$ *(resp. $\leq \sqrt{b}$),*

2. $\|d_z\angle(e_m(z), e_1(z))\| \leq Kb$ *(resp. $\leq \sqrt{b}$),*

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3. $\|T_z f^i(e_i)\| \leq (Kb)^i$ (resp. $\leq b^{i/2}$), for $1 \leq i \leq m$.

Note that we do not need the Collet-Eckmann condition to be satisfied, but only a projective hyperbolicity (*i.e.* dominated splitting) condition to be satisfied.

Binding Also we can imagine that if a point z satisfies such a projective hyperbolic condition, every point of a small plaque of $\mathcal{F}^m$ containing z should be projectively hyperbolic. Thus by Proposition 4.4, this plaque has a geometry close to a plaque of $\mathcal{F}_1$ and is exponentially fast contracted. Indeed, this enables us to have a better estimate on the projective hyperbolicity as pointed out by the following lemma:

Lemma 4.6 ([BC91] lemma 5.5, [YW01], lemma 2.2)**.** *Let $(M_j)_{j \geq 0}$ and $(M'_j)_{j \geq 0}$ be two sequences of 2×2 matrices of norms less than e^{2c^+} and with Jacobian less than $b \geq 0$ (which is small). Let $\kappa, \lambda \in \mathbb{R}$ such that $Kbe^{-2\kappa} < e^{-\lambda} < e^{-12c^+}e^{8\kappa}$. Let $p \geq 0$ be an integer such that*

$$d(M_j, M'_j) \leq e^{-\lambda j}, \quad \forall j \leq p,$$

and for some unit vector w,

$$\|M_j \circ M_{j-1} \circ \cdots \circ M_1(w)\| \geq e^{\kappa j}, \quad \forall j \leq p.$$

Then we have:

$$\frac{\|M'_j \circ M'_{j-1} \circ \cdots \circ M'_1(w)\|}{\|M_j \circ M_{j-1} \circ \cdots \circ M_1(w)\|} \geq \frac{1}{2}, \quad \forall j \leq p,$$

$$\angle\big(M'_j \circ \cdots \circ M'_1(w), M_j \circ \cdots \circ M_1(w)\big) \leq e^{-j\lambda/4},$$

with $\angle(u,v)$ the angle between u and v.

This lemma is particularly useful to forward the estimate from one point to a small plaque in $\mathcal{F}_p$ of it.

This phenomenon first appeared in [BC91] and is called *binding*. A point z *binds* a point z' for a time p if:

$$d(f^n(z), f^n(z')) \leq e^{-\lambda j - 2c^+}, \quad \forall j \in \{0, \ldots, p\}.$$

4.5   Common sequence of regular pieces

Definition 4.7. A *common sequence* of pieces from a flat stretched curves S is a sequence $(\alpha_i)_{i \geq 0}$ of regular puzzle pieces associated to flat curves $(S^{\alpha_0 \star \cdots \star \alpha_{i-1}})_i$ respectively, which satisfies the following properties:

1. Each piece is either e, either simple or either in $\mathcal{I}$,

2. the following inequality holds for every $j \geq 0$:

$$\sum_{l \leq j:\ \alpha_l \text{ is not simple}} n_{\alpha_l} \leq \mu \sum_{l=0}^{j-1} n_{\alpha_j},$$

3. $\alpha_i = e$ iff $i \in \{0, 1\}$; if α_n, α_{n+1}, ... and α_{n+k} are equal to s_2^+ then n is greater than $k20/M$.

By misuse of language, we call $c_i := \bigstar_{j=0}^i \alpha_j := \alpha_0 \star \alpha_1 \star \cdots \alpha_{i-1} \star \alpha_i$ a *common piece of depth i* and we write $c_i > c_{i+1}$ for every $i \geq 1$.

The following proposition associates to a common piece c_i a *common extension* $\{Y_{(c_i)}, \chi_{c_i}\}$.

Proposition 4.8. *Let $(\alpha_j)_j$ be a common sequence of pieces then the piece $c_i := \bigstar_{j=1}^i \alpha_j$ has a well defined extension $\{Y_{(c_i)}, \chi_{(c_i)}\} := (\cdots (\{Y_{\alpha_0}, \chi_{\alpha_0}\} \star \{Y_{\alpha_1}, \chi_{\alpha_1}\}) \cdots) \star \{Y_{\alpha_i}, \chi_{\alpha_i}\}$, with $\{Y_{\alpha_j}, \chi_{\alpha_j}\}$ the simple or canonical extension of α_j.*

The curves $\partial^u Y_{(c_i)}$ are two segments of the line $\{y = -2\theta\}$ and $\{y = 2\theta\}$ respectively. The curves $\partial^s Y_{(c_i)}$ consists of two segments of $\sqrt{b}$-C^1-close to plaques of $\mathcal{F}_1$.

Proof. The piece Y_{c_i} is included in $Y_e \setminus \mathcal{I}$ since it is included in Y_{α_2} which must be simple by the inequality of the common sequences' definition. Let us show this proposition by induction on i.

Let $i \geq 2$, suppose that $\alpha := c_i$ is well defined and put $\beta := \alpha_{i+1}$. The proof is easy when β is simple. Let us suppose that β is not simple, this implies that $\{Y_\beta, \chi_\beta\}$ is the canonical extension.

We suppose also by induction that for $2 \leq j < i$, for every $z \in Y_{c_j} \cap f^{-n_{c_j}}(Y_{\alpha_{j+1}})$, the map $T_z f^{n_{c_j}}$ sends $\chi_{c_j}(z)$ into $\chi_{\alpha_{j+1}}(f^{n_{c_j}}(z))$.

Lemma 4.9. *For every $z \in Y_{(c_j)}$, for every unit vector $u \in \chi_{(c_j)}(z)$, we have:*

$$e^{n_{c_j} c^-} \leq \|T f^{n_{c_j}}(u)\| \leq e^{n_{c_j} c^{++}}$$

Proof. The vector u is sent $T f^{n_{c_k}}$ into $\chi_{\alpha_{k+1}}$ for each $k < j$. By Proposition 2.1, each time that α_k is simple the expansion rate is between e^c and e^{c^+}. The proportion of non-simple pieces is less than μ and the expansion rate of them is between $e^{c/3}$ and e^{2c^+}. This is why we get the above estimate. $\square$

Using these inequalities, one proves that for every $n \geq n_{c_j} = n_\alpha$, for every $(z, u0 \in \chi_{c_j} = \chi_\alpha$:

(1) $$\|T_z f^n(u)\| \geq e^{-Mnc^+}$$

Therefore the geometry of the $\partial^s Y_{(c_j)}$ is C^1-close to arc of parabola by Corollary 4.5.

Markov and cone Properties. By (1), the $\mathcal{F}_{n_\alpha}$-leaves of points of S_α cover all $Y_{(\alpha)}$ but a proportion of the order of $(b)^{n_\alpha/2}$. Thus for every $z \in Y_\alpha$ and $u \in \chi(z)$ can be bound by some $(z', u') \in TS_\alpha$ for a times n_α with a factor $\sqrt{b}$. Thus the distance between $f^{n_\alpha}(z)$ and $f^{n_\alpha}(z')$ is smaller than $b^{n_\alpha/2}$. Also by binding, the angle between $T_z f^{n_\alpha}(u)$ and $T_{z'} f^{n_\alpha}(u')$ is smaller than $b^{n_\alpha/8}$. Using the flatness of the curves S' on which α is attached, we get that $T f^{n_\alpha}(u)$ belongs to χ_β if $f^{n_\alpha}(z)$ belongs to Y_β.

By the cone property, the set $Y_{(\alpha)}$ is homeomorphic to a square such that the curves corresponding to $[0,1] \times \{y\}$ have their tangent spaces $\chi_\alpha = \chi_{|Y_{(\alpha)}}$, for every $y \in [0,1]$. All these curves are sent by f^{n_α} to stretched curves. Moreover by the above argument, these curves are $b^{n_\alpha/2}$-C^0-distant to S'. By the mapping cone properties, they intersect $\partial^s Y_\beta$ transversally. Moreover by the Markov and cone property of β, this intersection is done at exactly one point of each curve of $\partial^s Y_\beta$, since

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ϵ_β is large with respect to $b^{n_\alpha/4}$. Consequently the operation $\{Y_\alpha, n_\alpha\} \star \{Y_\beta, n_\beta\}$ is well defined and satisfies the cone property with $\chi_{(\alpha \star \beta)}$ equal to the restriction $\chi_{|Y_{(\alpha \star \beta)}}$.

h-times property is obvious since f^{n_α} sends $\chi_{\alpha \star \beta}$ into χ_β.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We notice that $\left(Y_{(c_i)}\right)_i$ is a decreasing sequence of compact sets. Moreover, by h-property of the intersection of $Y_{(c_i)}$ with any horizontal line $\{y = t\}$ with $t \in [-2\theta, 2\theta]$ consists of segment of length less than $e^{-\frac{c}{3}n_{c_i}}$. Thus $W_c^s := \cap_{i \geq 1} Y_{(c_i)}$ is a Lipschitz curve which is $(\sqrt{b})$-close to an arc of parabola $\{y = x^2 + cst\}$ and its end points are in $\{y = x^2 + cst\}$ and its end points are in $\{y = -2\theta\} \cup \{y = 2\theta\}$. As f is of class C^2, by (1) on W_c^s for every $k \geq 2$, the curve W_c^s is actually of class C^2.

4.6 Atomic splitting

The leitmotiv of the strongly regular dynamics definition is the following situation.

Let S be a flat, stretched curve and $c = (\alpha_i)_i$ a common sequence of regular pieces from $\mathbb{S}$ such that $\hat{S} := f^M(f(S) \cap Y_{w_-^M})$ is tangent to W_c^s. We recall that $\mathbb{S}$ is the flat stretched curve equal to the half local unstable manifold $S^{(s_2^-)_{n \leq 0}}$ of $\mathcal{A}$.

The third property of the common sequence definition gives us a lower bound of the distance between the tangency point and the curves $\partial^s Y_{(c_j)}$ with $c_j := \bigstar_{i=1}^j \alpha_i$.

As all the curves $\partial^s Y_{(c_j)}$ $\sqrt{b}$-C^1-close to an arc of parabola and as $\hat{S}$ is C^2-close to the curve $[\mathcal{A}, f^{M+1}(0)] \cdot [f^{M+1}(0), \mathcal{A}]$, we deduce that $\hat{S}$ intersects $Y_{c_{j+1}} \setminus Y_{c_j}$ at zero or two curves, for $j \geq 1$. Let $\Delta := (c_j, c_{j+1})$. If this intersection is non empty, let S_{Δ_-} and S_{Δ_+} be the two backward images of these curves by $f^M_{|Y_{w_-^M}} \circ f_{|S}$, such that S_{Δ_-} is at the left of S_{Δ_+}.

We denote by $\Delta_\pm$ the pairs $\{S_{\Delta_\pm}, n_\Delta\}$, with $n_\Delta := M + 1 + n_{c_j}$. The pairs $\Delta_\pm$ are not puzzle pieces since $f^{n_\Delta}(S_{\Delta_\pm})$ does not stretch across Y_e. However we will prove that they satisfy the h-times property and that $f^{n_\Delta}(S_{\Delta_\pm})$ is flat. This will enable us to define new puzzle pieces of S as the $\star$-product of $\Delta_\pm$ with any puzzle pieces of $f^{n_{S_\Delta}}(S_{\Delta_\pm})$. Actually all the puzzle pieces that we will consider will be of the form $\Delta_\pm^1 \star \cdots \star \Delta_\pm^k \star s$, with $\Delta_\pm^i$ as above and s simple. This is why we call the pair Δ_+ and Δ_- *atomic pieces*.

Unfortunately $\check{S} := f^{n_\Delta}(S_{\Delta_\pm})$ is not stretched and so we would have to consider the puzzle pieces of curves that are not stretched. If we repeat the algorithm we may have to consider curves that do not intersect $\mathcal{R}_{\alpha^0}$ nor $\mathcal{R}_{\alpha^1}$. It is therefore quiet complicated to state the tangency condition. This is why we prefer to extend artificially the curve $\check{S}$ to a curve S^{Δ_+} (resp. S^{Δ_-}). For the parameter exclusion, it is useful to have $S^{\Delta_\pm}$ C^1-close to $\mathbb{S}^{c_j}$, in order that these two curves intersects the same common pieces of low depth. This is why we proceed as follows.[3]

We recall that S^{c_j} is a curve stretched and flat by regularity of c_j. Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$ be the function whose graph is S^{c_j}. The function ϕ is everywhere twice differentiable, with second

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[3]If one want to lose less parameter, he can make a dump between ϕ and ϕ_Δ at $Y_{\alpha_{j+1}}$. However, the scope of this manuscript is to the easiest possible to prove the main theorem.

derivative ϕ'' of norm less than θ. We suppose the existence of a function $\Psi : \ J \subset \mathbb{R} \to \mathbb{R}$ a.e. twice differentiable and whose graph is $\check{S}$ (this is actually always true).

$$\text{Let} \quad l: \ x \in \mathbb{R} \mapsto \left\{ \begin{array}{ll} \Psi''(x) & \text{if well defined} \\ \phi''(x) & \text{otherwise} \end{array} \right.$$

Let $x_0 \in J$ and $\Psi_{\Delta_\pm} := x \mapsto \int_{u=x_0}^{x} \left(\Psi'(x_0) + \int_{x_0}^{u} l(t)dt \right) du + \Psi(x_0)$.

Let $S^{\Delta_\pm}$ be the stretched curves equal to the intersection of Y_e with the graph of Ψ_Δ. We notice that $S^{\Delta_\pm}$ has θ-Lipschitz derivative if $\check{S}$ is flat. The curve $S^{\Delta_\pm}$ is called the Δ-*extension*.

Example 4.10. If S' is a segment of a horizontal line, then $S^{\Delta_\pm}$ is the C^1-curve made by $\check{S}$ and 0, 1, or 2 straight segments.

To complete the covering of S by simple pieces and atomic pieces, we shall add the following one for $\Delta := (e, e)$. Let S_{Δ_-} and S_{Δ_+} be the two components of $f_{|S}^{-1}(f(S) \cap Y_{w_-^{M-1}})$ at the left and the right respectively. We put also $\Delta_\pm := \{S_{\Delta_\pm}, n_\Delta := M + 1\}$ which are actually puzzle piece. We will proof that under that these hypotheses these puzzle pieces are regular.

We denote by $S^{\Delta_\pm}$ the stretched curves $f^{n_\Delta}(S_{\Delta_\pm})$. We will show that these curves are flat.

5 Definition of strongly regular dynamics

5.1 Statement

The dynamics f is strongly regular if:

- there exists an increasing sequence of puzzle algebras $(\Sigma_i, \mathcal{Y}_i)_{i \geq 1}$; we put $\Sigma' := \cup_i \Sigma_i$,

- there exists an increasing sequence of family of flat stretched curves $(\tilde{\Sigma}_i)_i$ such that for each $S \in \Sigma_i$ is endowed with an increasing sequence of family of regular puzzle pieces $(\mathcal{Y}_j(S))_{j \geq i}$; we put $\tilde{\Sigma} := \cup_i \tilde{\Sigma}_i$,

- for every $S \in \Sigma \cup \tilde{\Sigma}$, a common sequence of pieces $c(S) := (\alpha_i(S))_{i=0}^{k}$ from $\mathbb{S}$ with $\alpha_i \in \mathcal{Y}_{i-1}$ for $i > 1$. We put $c_n(S) = \bigstar_{i=1}^{n} \alpha_i(S)$, for $j \leq k$.

satisfying the following properties.

Geometrical Position For every $S \in \Sigma \cup \tilde{\Sigma}$,

- $f^M(f(S) \cap Y_{w_-^M})$ is included in Y_e and tangent to the local stable manifold $W_{c(S)}^s$.

For $i = 1$, $\tilde{\Sigma}_1$ is empty and $(\Sigma_1, \mathcal{Y}_1)$ is the simple puzzle algebra.

For $i > 1$:

- $\tilde{\Sigma}_i$ is formed by the well defined, flat curves of the form $S^{\Delta_1 \star \cdots \star \Delta_l}$ satisfying:

    - $S_{\Delta_1 \star \cdots \star \Delta_l}$ is not empty,
    - the curve S belongs to Σ_{i-1}, and $S^{\Delta_1 \star \cdots \star \Delta_{l-1}}$ belongs to $\tilde{\Sigma}_{i-1}$,
    - the atomic piece Δ_l is of the form $(c_{i_l}, c_{i_l+1})_\pm$ with $i_l \leq i - 2$.

- $\mathcal{Y}_i(S)$ for $S \in \tilde{\Sigma}_{i-1} \cup \Sigma_{i-1}$ consists of the simple pieces and the well defined and regular pieces of the form $\Delta_\pm \star \beta$, with $\beta \in \mathcal{Y}_i(S^{\Delta_\pm})$ with $\Delta = (c_j(S), c_{j+1}(S))$, for $j \leq i - 3$.

- Σ_i is the union of Σ_{i-1} with the set of the curves of the form $S^{\alpha \star \beta}$ with $S \in \Sigma_{i-1}$, $\alpha \in \mathcal{Y}_{i-1}(S)$ and β equal to e or a composition of simple pieces.

- $\mathcal{Y}_i(S)$ for $S \in \tilde{\Sigma}_i \cup \Sigma_i \setminus \tilde{\Sigma}_{i-1} \cup \Sigma_{i-1}$ consists of the simple pieces.

Remark 5.1. For every $S \in \Sigma' \cup \tilde{\Sigma}$, the set $\cup_i \mathcal{Y}_i(S)$ is formed by puzzle pieces with disjoint interior.

Remark 5.2. An induction easily shows that for every curve $S \in \Sigma_j$, each piece $\alpha \in \mathcal{Y}_j(S)$ is of the form $\Delta_1 \star \Delta_2 \star \cdots \star \Delta_k \star s$, with s simple and Δ_i atomic. This product is unique and called the *atomic decomposition.*

Remark 5.3. The curves of $\tilde{\Sigma}_i$ are the well defined, flat curves of the form $S^{\Delta_1 \star \cdots \star \Delta_l}$ such that $S_{\Delta_1 \star \cdots \star \Delta_l}$ is not empty, and there exists $j < i$ satisfying:

- the curve S belongs to Σ_j,

- $\Delta_m = (c_{i_m}, c_{i_m+1})_\pm$ with $i_m \leq m + j$ and c_{i_m} the i_m^{th} common piece associated to $S^{\Delta_1 \star \cdots \star \Delta_{m-1}}$ the for every $m \leq l$,

- $i_l \leq i - 2$ and $j + l \leq i$.

We note that $f^n(S_{\Delta_1 \star \cdots \star \Delta_l})$ is a non empty segment in $S^{\Delta_1 \star \cdots \star \Delta_l}$, with $n = n_{\Delta_1 \star \cdots \star \Delta_l}$. We will show that the segment is bounded by a hook at each of its end points that are not in $\mathcal{R}_{\pm \alpha^0}$.

5.2 k-Strongly regular dynamics

We will see that most of the strongly regular dynamics correspond to a fat Cantor set of parameter a. In order to evaluate the measure of this Cantor we need to work on a class of dynamics that correspond to finitely many intervals of parameter excluded. They are the *k-strongly regular dynamics.*

5.2.1 Critical position

Let S be a stretched and flat curve. Let c be equal to e or equal to a common piece (attached to $\mathbb{S}$). Let $(Y_{(c)}, \chi_c)$ be its common extension. The curve S is in a *nice position with respect to c* if $S' := f^M(f(S) \cap Y_{w_-^M})$ intersects both curves of $\partial^s Y_{(c)}$ or does not intersect the interior of $Y_{(c)}$. For
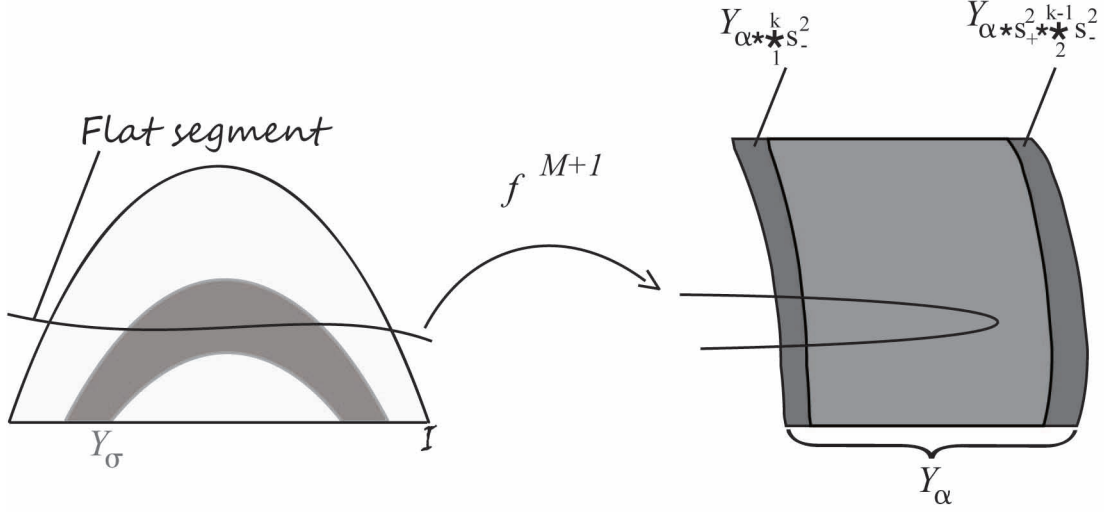
Figure 5: k-critical position of a good segment

$k \geq 1$, the curves S is in a *k-critical position with respect to* c if S is in a nice position with respect to only one of the pieces $c \star \bigstar_1^k s_-^2$ and $c \star s_+^2 \star \bigstar_1^k s_-^2$.

We notice that S'' is C^2-close to the concatenation $[\mathcal{A}, f^M(0,0)] \cdot [f^M(0,0), \mathcal{A}]$. Thus S'' must enter and exit in Y_α through only one curve of $\partial^s Y_\alpha$.

5.2.2 Statement

A dynamics f is *k-strongly regular* if there exist:

- an increasing family of puzzle algebras $(\Sigma_i, \mathcal{Y}_i)_{i=1}^k$,

- an increasing family of sets of flat stretched curves $(\tilde{\Sigma}_i)_{i=1}^k$ such that for each $S \in \Sigma_i$ is endowed with an increasing family of sets of regular puzzle pieces $(\mathcal{Y}_j(S))_{j=i}^k$,

- for every $S \in \Sigma \cup \tilde{\Sigma}$, a common sequence of pieces $c(S) := (\alpha_n(S))_n$ from $\mathbb{S}$ with $\alpha_i \in \mathcal{Y}_i$ for $i > 1$. We put $c_j(S) = \bigstar_{i=1}^j \alpha_i(S)$.

satisfying the following properties.

Geometrical Position   For every $S \in \Sigma_k \cup \tilde{\Sigma}_k$,

- Every $S \in \Sigma_k$ is in a $\left[\frac{M+j}{20}\right]$-critical position with $c_j(S)$, for $j \leq k$.

For $i = 1$, $\tilde{\Sigma}_1$ is empty and $(\Sigma_1, \mathcal{Y}_1)$ is the simple puzzle algebra.

For $k \geq i > 1$:

- $\tilde{\Sigma}_i$ is formed by the well defined, flat curves of the form $S^{\Delta_1 \star \cdots \star \Delta_l}$ satisfying:

    - $S_{\Delta_1 \star \cdots \star \Delta_l}$ is not empty,
    - the curve S belongs to Σ_{i-1}, and $S^{\Delta_1 \star \cdots \star \Delta_{l-1}}$ belongs to $\tilde{\Sigma}_{i-1}$,
    - the atomic piece Δ_l is of the form $(c_{i_l}, c_{i_l+1})_{\pm}$ with $i_l \leq i - 2$.

- $\mathcal{Y}_i(S)$ for $S \in \tilde{\Sigma}_{i-1} \cup \Sigma_{i-1}$ consists of the simple pieces and the well defined and regular pieces of the form $\Delta_{\pm} \star \beta$, with $\beta \in \mathcal{Y}_{i-1}(S^{\Delta_{\pm}})$ with $\Delta = (c_j(S), c_{j+1}(S))$, for $j + 2 < i$.

- Σ_i is the set of the curves of the form $S^{\alpha \star \beta}$ with $S \in \Sigma_{i-1}$, $\alpha \in \mathcal{Y}_{i-1}(S)$ and β equal to e or a composition of simple pieces.

- $\mathcal{Y}_i(S)$ for all $\alpha \in \tilde{\Sigma}_i \cup \Sigma_i \setminus \tilde{\Sigma}_{i-1} \cup \Sigma_{i-1}$ consists of the simple pieces.

Remark 5.4. $(\Sigma_i \cup \tilde{\Sigma}_i, \mathcal{Y}_i)$ is not a puzzle algebra since $\tilde{\Sigma}_i$ does not intersect the image of $(S, \alpha) \mapsto S^{\alpha}$.

Remark 5.5. For $S \in \Sigma_i \cup \tilde{\Sigma}_i$ and $\alpha \in \mathcal{Y}_j(S) \setminus \mathcal{Y}_{j-1}(S)$, we want to upper bound j in function of i and n_α. Let $\alpha = \Delta_1 \star \cdots \star \Delta_n \star s$ be the atomic decomposition of α. Let $\Delta_l := (c_{i_l}, c_{i_l+1})_{\pm}$ for every l.

The curve S^{Δ_1} belongs to $\tilde{\Sigma}_{j_1}$ if $j_1 - 2 \geq i_1$ and $j_1 \geq i + 1$. Thus S^{Δ_1} belongs to $\tilde{\Sigma}_{j_1}$ with $j_1 = i + i_1 + 2$. The curve $S^{\Delta_1 \star \Delta_2}$ belongs to $\tilde{\Sigma}_{j_2}$ if $j_2 - 2 \geq i_2$ and $j_2 \geq j_1 + 1 = i + i_1 + 2$. Thus $S^{\Delta_1 \star \Delta_2}$ belongs to $\tilde{\Sigma}_{j_2}$ with $j_2 = i + i_1 + i_2 + 2$ and so on $S^{\Delta_1 \star \cdots \star \Delta_n}$ belongs to $\tilde{\Sigma}_{j_n}$ with $j_n = i + \sum_l i_l + 2$.

We recall that $\mathcal{Y}_{j_n}(S^{\Delta_1 \star \cdots \star \Delta_n})$ contains the simple pieces s. Thus $\Delta_n \star s \in \mathcal{Y}_{j_n}(S^{\Delta_1 \star \cdots \star \Delta_{n-1}})$. And so on, the puzzle piece α belongs to $\mathcal{Y}_j \setminus \mathcal{Y}_{j-1}$ with $j \leq i + \sum_l i_l + 2 + n \leq i - M + n_\alpha + 1$ if α is not simple.

We notice that with $\mathcal{S}$ the set of (possibly empty) products of simple pieces:

$$\Sigma_{i+l} \supset \{S^{\alpha \star s} : S \in \Sigma_{i+l-1}, \ \alpha \in \mathcal{Y}_{i+l-1} \text{ and } s \in \mathcal{S}\} \supset \{S^{\alpha \star s} : S \in \Sigma_i, \ \alpha \in \mathcal{Y}_{i+l-1} \text{ and } s \in \mathcal{S}\}.$$

Consequently for all $S \in \Sigma_k \cup \tilde{\Sigma}_k$, the curve $\mathbb{S}^{c_k(s)}$ belongs to Σ_j with:

$$j - 1 \leq \sum_{i \leq k: \ \alpha_i(S) \text{ is not simple}} (2 + n_{\alpha_i(S)} - M) \leq \mu n_{c_k} \leq \mu M k.$$

6 Combinatorial Rigidity

For every $k \geq k$, every f k-strongly regular, the combinatorial definition of the algebra $(\Sigma_k, \mathcal{Y}_k)$ implies the following analytical properties. We call this principle *combinatorial rigidity*.

1. Every candidate for $\Sigma_i \cup \tilde{\Sigma}_i$ is well defined and flat.

2. Every candidate for $\mathcal{Y}_i$ is well defined and regular (if non empty).

3. Every puzzle pieces $\alpha \in \mathcal{Y}_i$ has its simple or canonical extension $\{Y_\alpha, \chi_\alpha\}$ that satisfies the following property:

   Dist For every flat curve $\gamma : [0,1] \to Y_\alpha$ from $x \in Y_\alpha$ to $y \in Y_\alpha$, such that $\partial_t \gamma(t)$ is a nonzero vector of $\chi_\alpha(\gamma(t))$ for every t, we have:

$$\left| \log \frac{\|\partial_t \gamma(1)\| \cdot \|T f^{n_\alpha} \partial_t \gamma(0)\|}{\|\partial_t \gamma(0)\| \cdot \|T f^{n_\alpha} \partial_t \gamma(1)\|} \right| \leq C(M) d\big(f^{n_\alpha}(x), f^{n_\alpha}(y) \big) \sum_{i=0}^{n_\alpha - 1} e^{-i\frac{c}{4}}.$$

   If moreover f is strongly regular:

4. For every $S \in \cup_i \Sigma_i$, the set of the points that do not belong to a piece of $\cup_i \mathcal{Y}_i(S)$ of order less than $n \geq M$ has Lebesgue measure less than $e^{-cn/4}$.

5. The set $\Sigma' := \cup_i \Sigma_i$ is endowed with the puzzle pieces $\mathcal{Y}' = \cup_i \mathcal{Y}_i$ compactifies to a regular structure $(\Sigma, \mathcal{Y})$ on the closure of the unstable set of the Horseshoe $\cup \Sigma_1$. In particular, the strongly regular dynamics are regular,

6. The SRB measure is unique, ergodic and physique (basin of positive Lebesgue measure).

By (1) and (2) the definition of strongly regular dynamics is indeed purely combinatorial and topological.

Let us prove these properties.

Throughout this section, f is assumed k-strongly regular.

6.1 Regular property of the puzzle pieces

The notion of critical point introduced in [BC91] is very useful for the estimate of lost of expansion in the critical strip $\mathcal{I}$.

Critical point Let S be a flat and stretched curve. A *critical point of order $m \geq 1$ of S* is a point $\tilde{z} \in \mathcal{I}$ of tangency between S and $\mathcal{F}_m$. This implies in particular that $\tilde{z}$ belongs to S and to the domain of $\mathcal{F}_m$.

We note that a f has possibly a critical set (in the common terminology), whose intersection with S is not necessarily contained in the critical set of S: however this is the case when the kernel of the differential of f contains the tangent space of S at these point. In particular, for $B = 0$ and $S = \mathbb{R} \times \{0\}$, the critical set of order k is the critical set of f^k intersected with S.

6.1.1 Estimate of the lost of expansion in the critical strip

The following proposition show how the critical points are useful.

Proposition 6.1. *Let $\tilde{z}$ be a critical point of order m of a flat curve S, satisfying the Collet-Eckmann condition $\mathcal{CE}^m$. Let $z \in \mathcal{I} \cap S$ and $w \in T_z S$. If there exists $p \in (0, m]$ such that every point of the segment $[z, \tilde{z}]$ of S satisfies $\mathcal{CE}^p$, then*

1. *There exists a splitting:*
$$w =: ue_p(z) + v(0,1)$$

   satisfying $|u| \leq (1 + Kb)\|w\|$ and $|v| \geq (1 - Kb)d\|w\|$, with d the distance between z and $\tilde{z}$.

2. *The norm $\|T_z f^p(w)\|$ is greater than $(1 - Kb)e^{c^- p}d\|w\| - b^p\|w\|$.*

Proof. The second statement is a simple consequence of the first by using the Collet-Eckmann condition $\mathcal{CE}^p$. Let us prove the first statement. By Corollary 4.5, for every $z' \in [\tilde{z}, z]$, the differential of the angle at z between e_p and e_1 is less than Kb:

$$\|T_z \angle(e_p, e_1)\| \leq Kb$$

The partial derivative with respect to x of the angle $e_p(x, y)$ and $T_{\tilde{z}}S$ is greater than $(1 - Kb)$. Since the segment $[z, \tilde{z}]$ has length d and is flat, the angle between $e_p(z)$ and $T_{\tilde{z}}S$ is greater than $(1 - Kb)d$. We conclude to the inequalities by using elementary trigonometry. $\qquad \square$

Let us prove the stated properties of the atomic pieces.

6.1.2 Critical pieces

Let $S \in \Sigma_{k-1} \cup \tilde{\Sigma}_{k-1}$, let $c_i =: c_i(S)$ be its associated common piece of depth $i \in \{1, \ldots, k\}$. Let S_{σ_i} be the preimage of $Y_{(c_i)}$ by $f^M_{|Y_{w^M}} \circ f_{|\mathcal{I}}$ intersected with S.

We remind that $Y_{(c_i)}$ is bounded by two segments $\partial^s Y_{(c_i)}$ of the lines $\{y = \pm 2\theta\}$ and by two curves $\partial^u Y_{(c_i)}$ that are (Kb)-C^1 close to arcs of parabolas. Therefore S_{σ_i} is a segment of S.

We call the pair $\sigma_i := \{S_{\sigma_i}, n_{\sigma_i}\}$ a critical piece, although it is not a puzzle piece since S_{σ_i} is not sent by $f^{n_{\sigma_i}}$ to a stretched curve.

For $i = 0$, we put $\sigma_0 := \{S \cap \mathcal{I}, M\}$.

In order to use Proposition 6.1, we need the following proposition.

Proposition 6.2. *Every point $z \in S_{\sigma_i}$ satisfies the Collet-Eckmann condition $\mathcal{CE}^{n_{\sigma_i}}$, for $i \leq k$. Moreover, we have:*
$$e^{c^- n_{c_i} + 2cM} \leq \|T_z f^{n_{\sigma_i}}(0,1)\| \leq e^{c^{++} n_{c_i} + 2c^+ M}$$

Proof. The inequality is an easy consequence of Lemma 4.9 and Proposition 2.1. Let us show the Collet-Eckmann property.

We have $\|T_z f(0,1) - 2(1,0)\| \leq b$. Also for $m \leq M$ the point $f^m(z)$ is outside of Y_e and $T_z f^m(0,1)$ is in χ. Thus:

$$\|T_z f^m(0,1)\| \geq e^{2cm - c}, \quad \forall m \in [1, M+1].$$

This proves the case $i \in \{0, 1\}$. Let us consider the case $i > 1$. By Proposition 2.1 we have also

$$T_z f^{M+1}(0,1) \in \chi \quad \text{and} \quad \|T_z f^{M+1}(0,1)\| \geq e^{2cM + c}.$$

Let n be the order of α_2. By Lemma A.2, the simple piece $Y_{s_\pm^n}$ are at most $2^{-n}/K$-close to $\{x = 0\}$. Consequently $T_{f^{M+1}(z)}f$ is at most $2^{-n}/K$ contracting. Thus we have

$$\|T_z f^{M+2}(0,1)\| \geq e^{2cM+c} \cdot \frac{2^{-M}}{K} \geq e^{c^-(M+2)}.$$

As for $j \in [M+2, M+1+n-1]$, $f^j(z)$ is outside of Y_e, we have:

$$\|T_z f^j(0,1)\| \geq e^{2cM+c} \cdot \frac{2^{-n}}{K} e^{(j-M-1)c} \geq e^{c^- j}.$$

And so on we prove the Collet-Eckmann condition and the inequality, using a similar argument as for Lemma 4.9. $\qquad\square$

We are now ready to prove the existence of a critical point in S_{σ_k}.

Proposition 6.3. *For $i \in \{0, \ldots, k\}$ there exists a unique critical point z_S^i in S_{σ_i} of order n_{σ_i} which satisfies $\mathcal{CE}^{n_{\sigma_i}}$. Moreover the image by f^{M+1} of z_S^i does not belong to $Y_{(c_i \star \bigstar_0^{i'+1} s_-^2)}$ and $Y_{(c_i \star s_+^2 \star \bigstar_0^{i'+1} s_-^2)}$, with $i' := \left[\frac{M+i}{20}\right]$, of order n_{σ_i}.*

Proof. For $i \in \{0, 1\}$, we take b small enough in order that the existing critical point for $B = 0$, at $0 \in \mathbb{R}^2$ persists as a critical point of order $n_{\sigma_i} = M + 1$ and is $K\theta$-close to 0.

Let $i > 1$. For any $z \in Y_{\sigma_i}$ the vector $(0,1)$ is sent by $T_z f^{M+1}$ to a nonzero vector of χ.

Since the points of $Y_{(c_i)}$ satisfy $\mathcal{CE}^{n_{c_i}}$, the foliation $\mathcal{F}_{n_{c_i}}$ is Kb-C^1-close to $\mathcal{F}_1$. Thus $T_z f^{M+1}(0,1)$ is not tangent to $\mathcal{F}_{n_{c_i}}$. In particular, f^{M+1} restricted to a neighborhood of S_{σ_i} is transverse to $\mathcal{F}_{n_{c_i}}$ and so the pull back $\mathcal{F}'$ of $\mathcal{F}_{n_{c_i}}$ by f^{M+1} restricted to a neighborhood of S_{σ_i} is well defined and of class C^2. From the geometric definition of the critical position, there exits a point of tangency between $f^{M+1}(S_{\sigma_i})$ and $\mathcal{F}_{n_\alpha}$. Such a point pullback to a tangency point ξ_0 between $\mathcal{F}'$ and S_{σ_i}. However, in general such a point is not a tangency point between $\mathcal{F}_{n_{\sigma_i}}$ and S. The point ξ_0 is $(Kb)^{n_{\sigma_i}}$-close to $S_{\sigma_i} \setminus f^{-M-1}(Y_{c_i \star \bigstar_1^{i'} s_-^2} \cup f^{-M-1}(Y_{c_i \star s_+^2 \star \bigstar_1^{i'} s_-^2})$, with $i' := [(M+i)/20]$. In order to give an upper bound of the angle between $T_{\xi_0}\mathcal{F}_{n_{\sigma_i}}$ and $T_{\xi_0}\mathcal{F}'$, we take two unit vectors e and e', tangent to these foliations and making this angle.

We split e' following the base $((0,1), e_{n_{\sigma_i}})$: $e' = u(0,1) + v e_{n_{\sigma_i}}$. We have $v \sim 1$ and $u \sim \angle(e', e_{n_{\Sigma_i}})$. By $\mathcal{CE}^{n_{\sigma_i}}$ we have:

$$Tf^{n_{\sigma_i}}(e')\| \geq |u| \cdot e^{n_{\sigma_i} c^-} - b^{n_{\sigma_i}}.$$

By the projective hyperbolicity $\mathcal{PCE}^{n_{\sigma_i}}$ of the common piece c_i, we have:

$$\|Tf^{n_{\sigma_i}}(e')\| \leq e^{2Mc^+} \cdot e^{2Mc^+}(e^{-Mc^+}b)^{n_{c_i}}.$$

Thus the angle $\angle(e, e')$ is less than $b^{n_{c_i}/2}$, since $n_{c_i} \geq 2$ and b is small with respect to e^{-M}.

The above angle is small with respect to the width of $Y_{(c_i \star s_+^2 \star \bigstar_1^{i'} s_-^2)} \setminus Y_{(c_i \star s_+^2 \star \bigstar_1^{i'+1} s_-^2)}$ and $Y_{(c_i \star \bigstar_1^{i'} s_-^2)} \setminus Y_{(c_i \star \bigstar_1^{i'+1} s_-^2)}$. Using the flatness of S and the lower bound of the derivative of $\angle(T\mathcal{F}_{n_{\sigma_i}}, (1,0))$ with respect to $(1,0)$ equal to 1, we conclude to the existence of z_S^i in the requested region.

As $\mathcal{CE}^{n_{\sigma_i}}$ is satisfied on S_{σ_i}, the function $z \in S_{\sigma_i} \mapsto \angle(e_{n_{\sigma_i}}, T_z S_{\sigma_i})$ is C^1-close to $z \in S_{\sigma_i} \mapsto \angle(e_1(z), T_z S_{\sigma_i})$ and so is monotone. This proves the uniqueness of the critical point. $\qquad\square$

Let us come back to the atomic pieces. Let $\Delta = (c_i, c_{i+1})$ with $i < k$.

The following estimate is weak and will be improved.

Lemma 6.4. *The distance between the critical point z_S^i and $S_{\Delta_+} \cup S_{\Delta_-}$ is greater than $e^{-(n_{\sigma_{i+1}} + 2[\frac{M+i+1}{20}])c^{++}}$.*

Proof. We simply use the upper bound e^{2c^+} for Tf in $\mathcal{D}$, and the upper bound e^{-Mc} for Tf along S at the first iteration, the upper bound of $e^{c^+ n_c}$ of Tf^{n_c} restricted to any common piece c, we then only need to compute the order of the pieces $c_i \star s_+^2 \star \bigstar_1^{i'+1} s_-^2$ and $c_i \star \bigstar_1^{i'+1} s_-^2$. $\qquad\square$

6.1.3   Well definition of candidates of $\mathcal{Y}_k$

For $S \in \Sigma_k \cup \tilde{\Sigma}_k \setminus (\Sigma_{k-1} \cup \tilde{\Sigma}_{k-1})$, the puzzle pieces of $\mathcal{Y}_k(S)$ are simple and so always well defined.

Let $S \in \Sigma_{k-1}$ and Δ equal to $(c_l(S), c_{l+1}(S))$ with $l+2 < k$. Let $\beta \in \mathcal{Y}_{k-1}(S')$ of the form $S^{\Delta\pm}$. We suppose that β belongs to $\mathcal{Y}_{k-1}(S')$ with S' of the form $S^{\Delta\pm}$.

We want to show that the pieces $\alpha := \Delta_\pm \star \beta$ are well defined and regular if $S_\beta := f_{|S_\Delta}^{-n_\Delta}(S'_\beta)$ is not empty. We remind that $n_\alpha := n_\Delta + n_\beta$.

The well definition of $\Delta_\pm \star \beta$ and its Markov property is a consequence of the following lemma:

Lemma 6.5. *The piece $\alpha_{l+1}(S)$ is equivalent a one piece $\alpha' \in \mathcal{Y}_k(S^{\Delta\pm})$. By equivalent we mean that the canonical extensions of $\alpha_{l+1}(S)$ and α' have their stable sides $\partial^s Y_{\alpha_{l+1}(S)}$ and $Y_{\alpha'}$ that intersect each other at both curves.*

Proof. By induction on k and by Proposition 6.15 (see below), the curve $\mathbb{S}^{c_l}$ and $S^{\Delta\pm}$ are $b^{n_{c_l}/8}$-C^1-distant. Thus they have the same first common pieces associated. This implies that their puzzle pieces of small order with respect to $\frac{n_{c_l}}{\theta}$ (as α_{l+1}) are equivalent.

This argument will be detailed in the proof of Proposition 6.15. $\qquad\square$

From this lemma we deduce that either S_β is included in f^{n_Δ} or have empty intersection. This shows that $\{S_\alpha, n_\alpha\}$ is well defined and satisfies the Markov property, whenever S_α is nonempty.

6.1.4 h-time Property

We prove by induction that every $\alpha \in \mathcal{Y}_k$ has its canonical extension that satisfies the following (strong) h-time property:

For every $(z, u) \in \chi_\alpha$, every $j \le n_\alpha$:

$$\|T_z f^{n_\alpha}(u)\| \ge e^{c(n_\alpha - j)/3} \|T_z f^j(u)\|.$$

The case $k = 1$ was done in Proposition 2.1 since $\mathcal{Y}_1$ consists only of simple pieces. Let $\alpha = \Delta_\pm \star \beta \in \mathcal{Y}_k$ with $\beta \in \mathcal{Y}_{k-1}$.

By Claim 4.2, for every $(z, u) \in \chi_\alpha$ is sen by Tf^{n_Δ} in χ_β. Thus, the h-time property of $\{Y_\alpha, \chi_\alpha\}$ is a consequence of the h-time property of $\{Y_\beta, \chi_\beta\}$ and of the following lemma:

Lemma 6.6. *For every $(z, w) \in \chi_\alpha$, every $m \le n_\Delta$ we have:*

$$\|T_z f^{n_\Delta}(w)\| \ge e^{(n_\Delta - m)c/3} \|T_z f^m(w)\|.$$

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Proof. In order to simplify the computation, we suppose that w is a unit vector.

Case $m = 0$. Let $\Delta = (c_l(S), c_{l+1}(S))$. By Proposition 6.2, we have that

$$\|T_z f^j(0,1)\| \geq e^{jc^-}, \quad \forall j \leq n_\Delta.$$

We split again $w = u(0,1) + v e_{n_\Delta}$, with $|u| \geq (1 - Kb)d(z, z_S^{l+1})$ and $v \leq (1 + Kb)$.

Hence $\|T_z f^{n_\Delta}(w)\| \geq |u|e^{n_\Delta c^-} - (1 - Kb)(be^{-c^-})^{n_\Delta}$.

Thus to solve the case $m = 0$, it is sufficient to provide a good lower bound of $|u| \geq (1 - Kb)d(z, z_S^{l+1})$. We need here a better estimate than before. To make lighter the notation, we denote by $\tilde{z} := z_S^{l+1}$. By $\mathcal{CE}^{n_\Delta}$ and binding, a plaque P of $\mathcal{F}_{n_\Delta}$ containing z is (Kb)-C^1-close to an arc of parabola and so intersects the vertical line passing through $\tilde{z}$ at a unique point z_v.

Let us parametrize the segment p of P between z and z_v and the segment γ of S between $\tilde{z}$ and the vertical projection $z - h$ of z into S as graphes of functions $t \mapsto p(t)$ and $t \mapsto \gamma(t)$ in the identification of Y_e with $[-1, 1] \times [2\theta, 2\theta]$. In such identification, $p(t)$ projects vertically to $\gamma(t)$.

As $\epsilon_\alpha = \theta^{n_\alpha}$ is small with respect to $d(\tilde{z}, S_\Delta) \geq e^{-(n_{\sigma_{l+1}} + 2[(M+l+1)/20])c^{++}}$, we have:

$$d(z_h, z) \leq \mu d(z, \tilde{z})^2.$$

$$\Rightarrow d(z_v, \tilde{z}) \leq \int_{t=0}^{d(z,\tilde{z})} |\dot{p}(t) - \dot{\gamma}(t)| + \mu d(z, \tilde{z})^2$$

$$\Rightarrow d(z_v, \tilde{z}) \leq \int_{t=0}^{d(z,\tilde{z})} \left|\angle(e_{n_\Delta}(z_v), T_{\tilde{z}}S)\right| + \int_{u=0}^{t} |\dot{p}(u) - \dot{\gamma}(u)| du\, dt + \mu d(z, \tilde{z})^2$$

By $\mathcal{CE}^{n_{\sigma_{l+1}}}$, the angle $|\angle(T_{\tilde{z}}S, e_{n_\Delta}(\tilde{z}))| = |\angle(e_{n_{\sigma_{l+1}}}, e_{n_\Delta}(\tilde{z}))| \leq (Kb)^{n_\Delta}$.

By $\mathcal{CE}^{n_\Delta}$, the derivative e_{n_Δ} on $[z_v, \tilde{z}]$ is Kb-small since Kb-close to the one of e_1.

Also, by $\mathcal{CE}^{n_\Delta}$ on $\{\gamma\}$, the vector field e_{n_Δ} is Kb-C^1-close to e_1. and so $|\dot{p}(u) - \dot{\gamma}(u)| \leq 1 + K\theta$.

$$d(z_v, \tilde{z}) \leq \int_{t=0}^{d(z,\tilde{z})} Kb\, d(z_v, \tilde{z}) + \int_{u=0}^{t}(1 + K\theta)du\, dt + \mu d(z, \tilde{z})^2$$

$$\Rightarrow d(z_v, \tilde{z}) \leq \frac{1 + K\theta}{2} d(z, \tilde{z})^2.$$

Consequently $d(z_v, \tilde{z})$ is less than $d(z, \tilde{z})^2$ or equivalently $d(z, \tilde{z})$ is greater than $\sqrt{d(z_v, \tilde{z})}$.

By similar argument as in Lemma 6.4, $d(z_v, \tilde{z})$ is greater than $e^{-n_{\sigma_{l+1}}c^+ - 2\left[\frac{M+l+1}{20}\right]c^+}$. Thus, the distance $d(z, \tilde{z})$ is greater than $e^{-\frac{1}{2}\left[2\left[\frac{M+l+1}{20}\right] + n_{c_{l+1}} + (2M+1)\right]c^+}$.

Consequently:

$$\|T_z f^{n_\Delta}(w)\| \geq (1 - Kb)e^{-\frac{1}{2}\left[2\left[\frac{M+l+1}{20}\right] + n_{c_{l+1}} + (2M+1)\right]c^+} \cdot e^{n_{c_l}c^- + (2M+1)c} - (1 + Kb)(be^{-c^-})^{n_\Delta} \geq e^{\frac{n_\Delta}{3}c}$$

Case $m \in \left(0, \max(M+1, n_\Delta/2)\right]$. If $\|T_z f^m(w)\| \leq K(Kb)^m$ then the h-time property is obvious. Otherwise, since $T_z f^m(w) = uT_z f^n(0,1) + vT_z f^m(e_{n_\Delta})$, we have:

$$\|T_z f^m(w)\| \leq e^{\frac{c}{2}} \|uT_z f^m(0,1)\|.$$

We recall that:

$$\|T_z f^{n_\Delta}(w)\| \geq (1 - Kb)\|uT_z f^m(0,1)\|.$$

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Thus in the case $m \in (0, M+1]$:

$$\frac{\|T_z f^{n_\Delta}(w)\|}{\|T_z f^m(w)\|} \geq (1 - Kb)e^{-\frac{c}{2}} \min_{U \in \chi(f^m(z)),\, \|U\|=1} \|T_{f^m(z)} f^{n_\Delta - m}(U)\|$$

$$\geq (1 - Kb)e^{\frac{c}{2} + cn_\Delta - m} \geq e^{\frac{c}{3}(n_\Delta - m)}.$$

In the case $m \in (M+1, n_\Delta/2]$, we have:

$$\|T_z f^{n_\Delta}(w)\| \geq (1 - Kb)|u|e^{n_{c_l}c + (2M+1)c}.$$

And:

$$\|T_z f^m(w)\| \leq e^{\frac{c}{2}}|u|\|T f^m_z(0,1)\| \leq e^{\frac{c}{2}}|u|e^{(2M+1)c^+ + (n-M-1)c^+}.$$

$$\Rightarrow \frac{\|T_z f^{n_\Delta}(w)\|}{\|T_z f^m(w)\|} \geq (1 - Kb)e^{\frac{c}{2}}e^{n_\Delta c - mc^+ + Mc - Mc^+} \geq e^{(n_\Delta - m)\frac{c}{3}}.$$

Case $m \in [\max\left(\frac{n_\Delta}{2}, M+1\right), n_\Delta]$.

Let $j < l$ be maximal such that $n := n_{\sigma_j} \leq \max(n_\Delta/2, M+1)$. The next proposition states that $T_z f^{n_{\sigma_i}}(w)$ belongs to the cone field of $\alpha_{i+1}(S)$, for every $i \geq j$. The h-properties of these pieces conclude the proof. $\qquad \square$

Proposition 6.7. *Let* $(z,w) \in \chi_\Delta$. *For every* $j < l$, *with* $n_{\sigma_j} \geq n_\Delta/3$ *we have:*

$$T_z f^{n_{\sigma_j}}(w) \in \chi_{\alpha_{j+1}} \quad \text{and} \quad \left|\angle\left(T_z f^{n_\Delta}(w), (1,0)\right)\right| \leq \sqrt{b}.$$

Proof. By binding we have:

$$\angle(T_z f^{n_{\sigma_j}}(0,1), T_{z'}S) \leq b^{n_{c_j}/8}$$

where z' is the image by $f^{n_{c_j}}$ of the intersection point of $\mathbb{S}_{c_j}$ with a $\mathcal{F}_{n_{c_j}}$-plaque of $f^{M+1}(z)$. By regarding the atomic decomposition of α_j with $c_j = \bigstar_i \alpha_i$, we have that $T\mathbb{S}^{c_j}$ is included in the cone centered at $(1,0)$ with the angle Kb by induction.

Then the proposition is a consequence of the following Lemma: $\qquad \square$

Lemma 6.8. *Let* $(z,w) \in \chi_\Delta$. *Then for every* $n \geq \frac{n_\Delta}{3}$, *we have:*

$$\angle(T_z f^n(w), T_z f^n(0,1)) \leq b^{n/2}$$

Proof. Let us write $w =: (u(0,1) + ve_n)\|w\|$, with $|u| \geq (1 - Kb)d(z, z_S^{l+1})$ and $|v| \leq (1 + Kb)$.
Thus:

$$|\angle(T_z f^n(w), T_z f^n(0,1))| \leq \frac{Kb^n}{d(z, z_S^{l+1}) \cdot e^{c^- n}} \leq Kb^n e^{-c^- n + 2n_{\sigma_{l+1}}c^+} \leq (Kb)^n e^{-c^- n + 12nc^+} \leq b^{n/2}$$

$\qquad \square$

6.1.5 *Dist*-property

We prove by induction on k that every piece $\alpha \in \mathcal{Y}_k$ have their canonical extension that satisfies the *Dist*-property. We notice that for $k = 1$, this property is satisfied since $\mathcal{Y}_1$ consists of simple pieces.

Let $k > 1$ and assume the induction hypothesis at step $k - 1$.

By the mapping property of the cone field, for any common piece c made by pieces of $\mathcal{Y}_{k-1}$ the extension $\{\mathcal{Y}_{(c)}, \chi_{(c)}\}$ satisfy the *Dist*-property by a similar computation as we did in the for the $\star$-product properties in subsection 3.1.

Let $\alpha = \Delta_\pm \star \beta \in \mathcal{Y}_k$. Let us show the Dist-property of $\{Y_\alpha, \chi_\alpha\}$.

First we recall that the cone field χ_α is sent by Tf^{n_Δ} in the cone field of χ_β.

By the h-property of $\Delta_\pm$, the *Dist*-property of $\{Y_\beta, \chi_\beta\}$ and the mapping property of the cone fields, the *Dist*-property of α is a consequence of the following Proposition:

Proposition 6.9. *For every flat C^2-curve $\gamma : [0, r] \to Y_\alpha$ such that its derivative $\partial_t \gamma$ is a unit vector in χ_α, we have:*

$$(2) \qquad D_1 := \left|\partial_t \log(\|T_{\gamma(t)} f^{n_\Delta}(\partial_t \gamma(t))\|)\right|_{t=0} \le D_0 := C \sum_{k=0}^{n_\Delta} e^{-ck/3} \left\|\partial_t\left(f^n \circ \gamma(t)\right)\right\|_{t=0}$$

Proof. To prove such an inequality, we construct a new vertical path β such that $\beta(t)$ is equal to the intersection point between the vertical line passing through $\gamma(0)$ with a plaque of $\mathcal{F}_{n_\Delta}$ containing $\gamma(t)$.

We note that β is a C^1 curve supported by the vertical line containing $\gamma(0)$. We prove Equation (2) in three steps.

First we compare D_1 with $D_2 := \left|\partial_t \log \|T_{\gamma(t)} f^{n_\Delta}(0, 1)\|\right|_{t=0}$. Then we prove that D_2 and $D_3 = \left|\partial_t \log(\|T_{\beta(t)} f^{n_\Delta}(0, 1)\|)\right|_{t=0}$ are equivalent. Finally we compare D_3 to D_0.

To evaluate D_1 we split as usual $w(t) := \partial_t \gamma(t) := u(t)(0, 1) + v(t) e_{n_\Delta}(\beta(t))$. We put $w_n(t) := T_{\gamma(t)} f^n(w(t))$ and $n = n_\Delta$. We note that:

$$D_1 = \frac{\partial_t \|w_n\|^2}{2\|w_n\|^2}(0) = \frac{< \partial_t w_n, w_n >}{2\|w_n\|^2}(0) = \frac{< \partial_t T_\gamma f^{n_\Delta}(u(0, 1)), w_n >}{2\|w_n\|^2}(0) + \frac{< \partial_t T_\gamma f^{n_\Delta}(v e_{n_\Delta}), w_n >}{2\|w_n\|^2}(0).$$

By using Cauchy-Lipschitz formula and dividing by $\|w_n\|^2 \sim \|T_\gamma f^{n_\Delta}(u(0, 1))\|^2 >> 1 >> \|T_\gamma f^{n_\Delta}(v e_{n_\Delta})\|^2$ (as in the proof of the h-time property), we get:

$$D_1 = \frac{\partial_t \|w_n\|^2}{\|w_n\|^2}(0) \le \frac{1 + Kb}{2} \left(\frac{\|\partial_t T_\gamma f^n(u(0, 1))\|}{\|T_\gamma f^n(u(0, 1))\|} + \frac{\|\partial_t T_\gamma f^n(v e_n)\|}{\|w_n\|} \right)(0)$$

Let us study the second term of this sum:

$$\frac{\|\partial_t T_\gamma f^n(v e_n)\|}{\|w_n\|}(0) \le |\partial_t v(t)| \frac{\|T_\gamma f^n(e_n)\|}{\|T_\gamma f^n(w)\|}(0) + |v(t)| \cdot \frac{\|\partial_t T f^n(v e_n)\|}{\|T_\gamma f^n(w)\|}(0).$$

By the h-time property $\|T_\gamma f^n(w)\| \ge e^{-\frac{c}{3} n}$ and $\|T_\gamma f^n(e_n)\| \le b^n$. Also $|v|(t) = \frac{\|w(t) \times e_n\|}{\|(0, 1) \times e_n\|}$ is a C^1-function bounded by K. Thus:

$$\frac{\|\partial_t T_\gamma f^n(ve_n)\|}{\|T_\gamma f^n(w)\|} \leq Kb^n e^{-\frac{c}{3}n} + Ke^{-\frac{c}{3}n}\|\partial_t T f^n(e_n)\|.$$

Consequently by the following lemma D_1 is less than $(1 + Kb)D_2 + b$:

Lemma 6.10 (Coro. 2.2 [YW01]). *Let $K_0 > 0$ and let $b > 0$ be a small. Let $(t \mapsto M_i(t))_i$ be a sequence of 1-parameter family of matrices with norms less than K_0 and Jacobean less than b. Let us suppose than $\|\partial_t M_i\| \leq K_0^i$ and $|\partial_t \det M_i| \leq K_0^i b$. If for some $\kappa >> \sqrt{b}$ we have $\|M^{(i)}(t)\| \geq \kappa^i$, for every $i \leq n$, with $M^{(i)} = M_i \times \cdots \times M_0$, then $\|\partial_t M^{(n)}(e_n)\| \leq \left(\frac{Kb}{\kappa^3}\right)^n$, with e_n the most contracted vector of $M^{(n)}$.*

On the other hand D_3 might be seen as the sum of the distortion of f^{M+1} along β (which is less than $C/2 = C(M)/2$) with the distortion of the regular (common) piece (which is less than $C \sum_{i=0}^{n-M-1} e^{-ic/3}$). By using the Collet-Eckmann property, we get that the distortion D_3 is less than $\frac{C}{2} + e^{(-2M-1)c}C\left(\sum_{j=1}^{n-M-1} e^{-jc/3}\right)\|\partial_t(f^n \circ \beta)(t)\|$.

Also $\|\partial_t(f^{n_\Delta} \circ \beta)(t)\|$ is equivalent to $\|T_\beta f^n(w)(t)\|$. By using the binding estimate of Lemma 4.6, we note that $\|T_\beta f^n(w)(t)\|$ is equivalent to $\|T_\gamma f^n(w)(t)\|$. Thus we get that D_3 is small with respect to D_0.

Thus it is sufficient to prove that D_2 and D_3 are equivalent. In order to do so we need the following extra lemma shown in the appendix:

Lemma 6.11. *Let $\nu \leq 1/10$ and $\sigma > 0$. Let $(M_i)_i$ and $(M'_i)_i$ be two sequences of the matrices of $\mathcal{M}_2(\mathbb{R})$ with norm less than K_0 and with Jacobian less than b, such that for every $i \leq n$:*

- $b \leq \frac{\sqrt{\nu}K_0^2}{10}$, $\nu \geq b$, $K_0 \geq 2$, *and* $r := (\nu K_0^{12})^{\frac{1}{4}} < 1/10$.

- $\|M_i - M'_i\| \leq \sigma \cdot \nu^{i-1}$,

- *there exists a unit vector V s.t. $\|M^{(i)}V\| \geq 1$ and $\|M'^{(i)}V\| \geq 1$, with $M^{(i)} := M_i \times \cdots \times M_1$ and $M'^{(i)} := M'_i \times \cdots \times M'_1$.*

Then we have $1 - \sigma\nu^{n/2} \leq \frac{\|M'^{(n)}(w)\|}{\|M^{(n)}(w)\|} \leq 1 + \sigma$.

We apply it with $V = (0, 1)$, $M_i = T_{f^i \circ \gamma(t)}f$, $M'_i = T_{f^i \circ \beta(t)}f$, $\nu = \sqrt{b}$, $\sigma := 3t$. We get:

$$D_2 \leq \limsup_{t \to 0} \frac{\|M^{(n)}(t)V - M^{(n)}(0)V\|}{t\|M^{(n)}(0)V\|} \leq \limsup_{t \to 0} \frac{\|M^{(n)}(t)V - M'^{(n)}(t)V\|}{t\|M^{(n)}(0)V\|} + D_3.$$

Also:

$$\left\|M^{(n)}V - M'^{(n)}V\right\|^2(t) \leq \left|\|M^{(n)}V\| - \|M'^{(n)}V\|\right|^2(t) + K\frac{\|M'^{(n)}(V) \times M^{(n)}(V)\|^2}{\|M'^{(n)}(V)\|\|M^{(n)}(V)\|}(t)$$

By Lemma 6.11:

$$\frac{\left|\|M'^{(n)}V\| - \|M^{(n)}V\|\right|}{\|M^{(n)}V\|}(t) \leq |t|.$$

By Lemma 6.16 (see below),

$$\frac{\left\|M'^{(n)}(V) \times M^{(n)}(V)\right\|}{\left\|M'^{(n)}(V)\right\|\left\|M^{(n)}(V)\right\|}(t) \leq \theta|t|.$$

Consequently, by the Collet-Eckamnn Property: $D_2 \leq D_3 + C/2$.

$\square$

6.2 Geometrical properties of Σ_k

We must explain the canonical compactification of $\cup_i \Sigma_i$. For this end let us define the metric of this space.

6.2.1 Transversal metric on Σ_k and canonical compactification

We define inductively T_k and $\tilde{T}_k$ which parametrize Σ_k and $\tilde{\Sigma}_k$. We recall that given presequences $\alpha := (t_i)_{i=-N}^0$ and $\beta := (t'_i)_{i=-m}^0$, with $N \in \mathbb{N} \cup \{\infty\}$ and $m \in \mathbb{N}$, the concatenation $\alpha \cdot \beta$ is the presequence $(t''_i)_{i=-N-m}^0$ with $t''_i = t'_i$ if $i \geq -m$ and $t''_i = t_{i+m}$ if $i < -m$.

We recall that the space T_1 of (infinite) presequences $(s_i)_{i \leq 0}$ of simple pieces parametrizes Σ_1.

As $\tilde{\Sigma}_1$ is empty, we put $\tilde{T}_1 := \emptyset$.

For $i > 1$, let us define:

$$T_i = \{t \cdot \alpha \cdot s;\ t \in T_i,\ \alpha \in \mathcal{Y}_{i-1}(S^t) \text{ and } s \text{ a composition of simple pieces}\},$$

$$\tilde{T}_i = \tilde{T}_{i-1} \cup \{t \cdot \alpha;\ t \in T_{i-1} \cup \tilde{T}_{i-1} \text{ and } \alpha = (c_j(S^t), c_{j+1}(S^t))_{\pm},\ j \leq i-2 \text{ and } S^\alpha \in \tilde{\Sigma}_i\}.$$

We remark that the maps $t \cdot \alpha \cdot s \in T_i \mapsto (S^t)^{\alpha \star s} \in \Sigma_i$ and $t \cdot \alpha \in (S^t)^\alpha \in \tilde{\Sigma}_i$ are bijection, thought the splitting $t \cdot \alpha$ and $t \cdot \alpha \cdot s$ are not unique.

Let us now put a distance on the spaces of presequences T_i and $\tilde{T}_i$. For this end we regard the spaces D_i and $\tilde{D}_i$ of finitely truncated elements of T_i and $\tilde{T}_i$ resp.

In other words, $D_i = \{(t_i)_{-n \leq i \leq 0} :\ n \geq 0,\ t = (t_i)_{i \leq 0} \in T_i\} \cup \{e\}$ and $\tilde{D}_i = \{(t_i)_{-n \leq i \leq 0} :\ n \geq 0,\ t = (t_i)_{i \leq 0} \in \tilde{T}_i\} \cup \{e\}$, where e denote the empty presequence.

We notice that for each $\alpha = (t_i)_{i=-n}^0 \in D_i \cup \tilde{D}_i$, we can define $n_\alpha = \sum_i n_{t_i}$ since each t_i is either a puzzle piece or an atomic piece. We are going to define the right division $/$ and the equivalence $\sim$ on $D_i \cup \tilde{D}_i$, such that $\alpha \mapsto n_\alpha$ is constant on the equivalence classes. Then the distance on $T_i \cup \tilde{T}_i$ will be $d(S^t, S^{t'}) = b^{\frac{t \wedge t'}{8}}$, with $t \wedge t'$ the order of a maximal common divisor of t and t'.

Let us now define the division $/$ by induction on i.

We write $t \sim t'$ and say that t and t' are *equivalent* iff t/t' and t'/t for $t, t' \in D_i \cup \tilde{D}_i$.

As we already did many times, every simple piece s_-^n (resp. s_+^n) attached to some curves S is equivalent to s_-^n (resp. s_+^n) attached to any other curves.

Let $\alpha, \beta \in D_1$. We have $\alpha = (s_{-n}, \cdots, s_0)$ and $\beta = (s'_{-m}, \cdots, s'_0)$ for simple pieces $(s_j)_{j=-n}^0$ and $(s'_j)_{j=-m}^0$. We say that α *divides* β if $m \geq n$ and $s_{-j} \sim s'_{-j}$ for all $j \leq n$.

If α divides β, we note β/α.

Let us suppose $/$ and so $\sim$ constructed on D_{i-1} and $\tilde{D}_{i-1}$ for some $i > 1$.

Let $\alpha, \beta \in D_i \cup \tilde{D}_i$. We say that α divides β if one of the following situations occur.

- The presequences β, α belong to $D_{i-1} \cup \tilde{D}_{i-1}$ and the pair (α, β) satisfies the rule given by the induction.

- The presequence β belongs to $\tilde{D}_i \setminus (D_{i-1} \cup \tilde{D}_{i-1})$ and (α, β) is of the form:

  - $\alpha = \alpha_1 \cdot \alpha_2$ and $\beta = \beta_1 \cdot \beta_2$ with $\beta_1, \alpha_1 \in D_{i-1} \cup \tilde{D}_{i-1}$ and β_1/α_1, and also α_2, β_2 are both equal to a same oriented pair $(c_j, c_{j+1})_+$ or both equal to a same oriented pair $(c_j, c_{j+1})_-$, possibly not attached to a same curve,

  - β is of the form $\beta' \star (c_l, c_{l+1})_\pm$ with c_l/α and $c_l, \alpha \in D_{i-1}$.

To state the last possibility, let us define $/$ and so $\sim$ on the puzzle pieces of $\mathcal{Y}_i|\Sigma_{i-1}$. By the atomic decomposition, all pieces $\alpha, \beta \in \mathcal{Y}_i|\Sigma_{i-1}$ can be written in the form $\Delta_{-n} \star \cdots \star \Delta_0 \star s$ and $\Delta'_{-m} \star \cdots \star \Delta'_0 \star s'$ with s, s' simples and with $\Delta_{-n}, \cdots, \Delta_0$ and $\Delta'_{-m}, \cdots, \Delta'_0$ in $\tilde{D}_{i-1}$. We put $\alpha \sim \beta$ if $n = m$ and $\Delta_j \sim \Delta'_j$ for every $j \geq -n$ and. We put α/β if $m \leq n$ and $\Delta_j \sim \Delta'_j$ for every $j > -m$, Δ_m/Δ'_m and $s \sim s'$.

We remark that in such settings the puzzle pieces are equivalent to their atomic decomposition.

- The presequence β belongs to $D_i \setminus (D_{i-1} \cup \tilde{D}_{i-1})$, α belongs to D_i and they are of the form:

  - $\beta = \beta_1 \cdot \beta_2 \cdot s$ and $\alpha = \alpha_1 \cdot \alpha_2 \cdot s$, with β_1/α_1 both in D_{i-1}, $\alpha_2 \sim \beta_2 \in \mathcal{Y}_i|\Sigma_{i-1}$, and s a finite composition of simple pieces,

  - $\beta = \beta_1 \cdot \beta_2 \cdot s$ and $\alpha = \alpha_2 \cdot s$, with β_2/α_2 both in $\mathcal{Y}_i|\Sigma_{i-1}$, and s a finite composition of simple pieces.

  - $\beta = \beta_1 \cdot \beta_2 \cdot s$ and s/α, with $\beta_1 \in \mathcal{D}_{i-1}$, $\beta_2 \in \mathcal{Y}_i|\Sigma_{i-1}$ and α, s product of simple pieces.

Remark 6.12. If $\alpha \sim \beta$ then $n_\alpha = n_\beta$.

Remark 6.13. A presequence $\alpha \in D_i$ may divides $\beta \in \tilde{D}_i$ whereas a presequence $\alpha \in \tilde{D}_i$ never divides $\beta \in D_i$, if $\alpha \neq e$.

Remark 6.14. We can identify the $\star$-product with the concatenation.

The *greatest common divisor* n_d of $(\alpha, \beta) \in (T_k \cup \tilde{T}_k)^2$ is the maximal order of d such that $d|\alpha_1$ and $d|\beta_1$, with $\alpha = \alpha_2 \star \alpha_1$ nd $\beta = \beta_2 \star \beta_1$. We denote $n_d = \alpha \wedge \beta$.

The distance on $T_k \cup \tilde{T}_k$ is $d(S^t, S^{t'}) = b^{\frac{t \wedge t'}{8}}$.

Let us prove by induction on k the following proposition useful for the parameters section and the compactification.

Proposition 6.15. *For f k-strongly regular, the following properties are satisfied:*

(a) *Let $S \in \Sigma_{k-1}$ and $\Delta := (c_i(S), c_{i+1}(S))$ for $i < k$. Then each extension $S^{\Delta\pm}$ is well defined, flat and stretched.*

(b) The function $t \in T_k \cup \tilde{T}_k \mapsto \mathbb{S}^t$ is Lipschitz for the C^1 topology,

Proof. **(a)** By Proposition 6.7, the tangent spaces of $f^{n_\Delta}(S_{\Delta_\pm})$ makes an angle less than $\sqrt{n}$ with $(1,0)$. Also the h-property of $\Delta_\pm$ implies by Lemma 3.3 that $f^{n_\Delta}(S_{\Delta_\pm})$ is flat. Thus the extension S^Δ is well defined and stretched.

As $\mathbb{S}^{c_i(S)}$ has curvature less than θ^2 by the h-property of $S_{\Delta_\pm}$ and Lemma 3.3, the angle between TS^Δ and $(0,1)$ is less than $\sqrt{b} + K\theta^2 << \theta$. Hence S^Δ is flat.
(b) The proof proceeds by induction on k.

To prove the case $k = 1$, it is sufficient to prove that if S and S' are two flat and stretched curves, then:

$$d(S^s, S'^s) << b^{n_s/8} d(S, S').$$

As the point of the extension Y_s satisfy $\mathcal{CE}^{n_s}$, we may foliate a proportion larger than $b^{n_s/2}$ of Y_s by $\mathcal{F}_{n_s}$-plaques of points in S_s. As Y_s is more than e^{-Mc^+}-distant to $\{x = 0\}$, these curves make an angle larger than e^{-Mc^+} with the vertical. Consequently, by flatness of S and S', the C^0-distance between S^s and S'^s is smaller than $b^{n_s/2}$ times the distance between S and S'. By using this binding argument with Lemma 6.11, we get that the C^1-distance between S^s and S'^s is small with respect to $b^{\frac{n_s}{8}} d(S, S')$.

Let $k > 1$, we consider $S^t, S^{t'} \in \Sigma_k \cup \tilde{\Sigma}_k$ which are not both in $\Sigma_{k-1} \cup \tilde{\Sigma}_{k-1}$. Using the previous step, it remains only two case to study (thanks to the atomic decomposition).

1) Case $S^t = S^{\Delta+}$ (resp. $S^t = S^{\Delta-}$) and $S^{t'} = S'^{\Delta+}$ (resp. $S^t = S^{\Delta-}$) with $\Delta = (c_i(S), c_{i+1}(S)) = (c_i(S'), c_{i+1}(S'))$ and $S, S' \in \Sigma_{k-1} \cup \tilde{\Sigma}_{k-1}$.

Let us prove that $S^{t'}$ is $b^{t \wedge t'/8}$-C^0-close to S^t.

By Proposition 6.2, every $z \in S_{\Delta_\pm} \cup S'_{\Delta_\pm}$ satisfies the Collet-Eckmann condition $\mathcal{CE}^{n_\Delta}$. By binding, local stable manifolds of the end points of $S_{\Delta_\pm}$ and $S'_{\Delta_\pm}$ are Kb-C^1-close to arcs of parabolas. By the critical position, the end points of these segment are at least $e^{-2c^+ + Mn_\Delta}$-distant to the top of these parabolas (the lower bound is large). Thus, by Proposition 4.4, most of the points of $S'_{\Delta_\pm}$ can be linked to a point of $S_{\Delta_\pm}$ via a small plaque P of $\mathcal{F}_{n_\Delta}$. Let us upper bound the proportion which is not linked and compare the length l of these plaques to the C^0-distance d between S and S'.

In order to do so, we are going to estimate the angle between the tangent space of these $\mathcal{F}_{n_\Delta}$-plaques and the tangent space of S. By Corollary 4.5, $\|d_z \angle(e_{n_\Delta}(z), e_1(z))\| \leq Kb$, for every $z \in P$.

By Lemma 6.4, the critical points z_S^i and $z_{S'}^i$ are more than $e^{-((1+\mu)n_\Delta + M + 2[(M+i+1)/20])c^{++}}$-distant to $S_{\Delta_\pm}$ and $S'_{\Delta_\pm}$ respectively. Consequently the angle between the tangent space of the plaques of P and TS is large with respect to $e^{-3n_\Delta c^+}$. Thus the plaques of $\mathcal{F}_{n_\Delta}$ link all but a proportion less than $(Kb)^{n_\Delta} e^{3n_\Delta c^+}$ of S_Δ and S'_Δ.

Also the length l is less than $de^{3n_\Delta c^+}$, with d the C^1-distance between S and S'. Also the Hausdorff distance between $\check{S} := f^{n_\Delta}(S_{\Delta_\pm})$ and $\check{S}' := f^{n_\Delta}(S'_{\Delta_\pm})$ is less than $d(Kb)^{n_\Delta} << db^{n_\Delta/2}$. Let z and z' be the end points of these curves that are not in $\mathcal{R}_{\pm\alpha_1}$.

For the same reason, the distance between z and z' is smaller than $de^{3n_\Delta c^+} b^{n_\Delta} << db^{n_\Delta/8}$.

To evaluate the angle $\angle(T_z S^t, T_{z'} S^{t'})$, we start by evaluating the angle $\angle(T_{z_0} S, T_{z_0'} S')$, where $z_0 \in S$ and $z_0' \in S'$ are sent by f^{n_Δ} to z and z' respectively. In the identification of Y_e with $[-1,1] \times [-2\theta, 2\theta]$, the vertical projection z_0'' of z_0 into S' satisfies:

$$\begin{cases} d(z, z_0'') \leq Kd \\ \angle(T_{z_0} S, T_{z_0''} S') \leq Kd \end{cases}$$

by flatness of the curve. Also z_0' and z_0'' are as close as $Ke^{3n_\Delta c^+} d$. Since S' is flat, we get:

$$\angle(T_{z_0'} S', T_{z_0''} S') \leq e^{3n_\Delta c^+} d$$

$$\Rightarrow \angle(T_{z_0} S, T_{z_0'} S') \leq Ke^{3n_\Delta c^+} d.$$

The following lemma implies that the angle $\angle(T_z S^t, T_{z'} S^{t'})$ is small with respect to $db^{n_\Delta/4}$.

Lemma 6.16. *Let $\nu \leq 1/10$, let $\sigma > 0$ and let $(M_i)_i$ and $(M_i')_i$ be two sequences of the matrices of $\mathcal{M}_2(\mathbb{R})$ with norm less than K_0 and Jacobian less than b, such that:*

- $b \leq \frac{\sqrt{\nu} K_0^2}{10}$, $\nu \geq b$, and $K_0 \geq 2$,

- $\|M_i - M_i'\| \leq \sigma \cdot \nu^{i-1}$,

- $|\angle(w, w')| \leq \sigma$,

- $\|M^{(n)} w\| \geq 1$, $\|M'^{(n)} w'\| \geq 1$, with $M^{(n)} := M_n \times \cdots \times M_1$ and $M'^{(n)} := M_n' \times \cdots \times M_1'$.

Then $|\angle(M^{(n)} w, M'^{(n)} w')| \leq 2\sigma(K_0^4 \nu)^{n/2}$.

This lemma will be shown in the appendix.

Let us now see that the C^1-distance between S^t and $S^{t'}$ is smaller than $db^{n_\Delta/8}$.

Also the distance between the end points z and z' of $\check{S}$ and $\check{S}'$ is small with respect to

$$db^{n_\Delta/4}$$

. Thus the $\theta db^{n_\Delta/8}$-neighborhood of $\check{S}$ and $\check{S}'$ in S^t and $S^{t'}$ respectively are still $\sqrt{\theta} db^{n_\Delta/8}$ close to each other for the C^1-distance. Let $\hat{S}$ and $\hat{S}'$ be in two such neighborhoods, such that their respective end points $\hat{z}, \hat{z}' \notin \mathcal{R}_{\pm a^1}$ have the same x-coordinate, are $\sqrt{\theta} b^{t \wedge t'/8}$-distant and the angle between $T_{\hat{z}} \hat{S}$ and $T_{\hat{z}'} \hat{S}'$ is less than $\sqrt{\theta} db^{n_\Delta/8}$. By definition of the extensions $S^t := S^\Delta$ and $S^{t'} := S'^\Delta$ for every pair of point $y \in S^t \setminus \check{S}$ and $y' \in S^{t'} \setminus \check{S}'$ with same x-coordinate, the angle between $T_y S^t$ and $T_{y'} S^{t'}$ is equal to the one between $T_{\hat{z}} \hat{S}$ and $T_{\hat{z}'} \hat{S}'$ and so is less than $\sqrt{\theta} db^{n_\Delta/8}$. The distance between y and y' is dominated by the distance between the x-coordinate of y and y' times the angle between $T_{\hat{z}'} \hat{S}'$ and $T_{\hat{z}} \hat{S}$. Consequently, the distance between y and y' is less than $db^{n_\Delta/8}$.

2) The last case is S^t of the form $S^{\Delta \pm}$ and $S^{t'} = S'^\alpha$ with $\Delta = (c_i(S), c_{i+1}(S))$, α/c_i and $S, S' \in \Sigma_{k-1}$. As all the inequalities are large, by using the triangular inequality we only need to prove that the C^1-distance between S^t and $S^{t'}$ is small with respect to $d(S, S') b^{n_\Delta/8}$, in the case $\alpha \sim c_i(S)$.

The C^0-distance is obtained as in case 1 by replacing s by the common piece $c_i(S)$. We recall that $f^{M+1}(S^{\Delta\pm})$ is included in $Y_{(c_i(S))}$. The foliation $\mathcal{F}_{n_{c_i(S)}}$ restricted to $Y_{c_i(S)}$ is Kb-C^1-close to $\mathcal{F}_1$. Also $\partial^s Y_{c_i(S)}$ is $(Kb)^{n_{(c_i(S))}}$-C^1-close to plaques of $\mathcal{F}_{n_{c_i(S)}}$. Consequently all $f^{M+1}(S^{\Delta})$ but a proportion less than $(Kb)^{n_{c_i(S)}}$ is included in $\mathcal{F}_{n_{c_i(S)}}$-plaques of points of $\mathbb{S}_{c_i(S)}$. Thus $f^{n_{\Delta}}(S_{\Delta\pm})$ is included in the $b^{n_{c_i(S)}/8}$-neighborhood of $\mathbb{S}^{c_i(S)}$. By Lemma 6.8 for $z \in S_\Delta$, the tangent space at $f^{n_\Delta}(z)$ of $S^{\Delta\pm}$ is $b^{n_\Delta/2}$-close to $T_z f^{n_\Delta}(0,1)$. By Lemma 4.6 $T z f^{n_\Delta}(0,1)$ is $b^{g'/7}$-close to the tangent space of $S^{t'} = S'^\alpha$. As in case 1, we finish the proof of the C^1-estimates.

$\square$

6.2.2   Hooks loci

Let $S \in \tilde{\Sigma}_k$ be of the form S'^β with $S' \in \Sigma_i$ and β a composition of atomic pieces. We are going to show that $f^\beta(S'_\beta)$ is bounded by hooks or by rays $\mathcal{R}_{\pm\alpha^0}$. This will enable us to associate injectively to S a slice of $f^{n_\beta}(S')$. *This will be crucial to upper bound the cardinality of $\check{\Sigma}_k$.*

Let β be the product $\Delta_1 \star \cdots \star \Delta_j$, with Δ_i of the form $(c_{n_i}, c_{n_i} \star \beta_i)_\pm$ and with c_{n_i} the n_i^{th} common piece associated to $S'^{\Delta_1 \cdots \Delta_{i-1}}$. Let $\beta_i := \Delta_1^i \star \cdots \star \Delta_{m_i}^i \star s_i$ be its atomic decomposition. Let $\beta_i^{(l)} := \Delta_{l+1}^i \star \cdots \star \Delta_{m_i}^i \star s_i$.

Proposition 6.17. *Each end of $\gamma_j := f^n(S_{\Delta_1 \star \cdots \star \Delta_j})$ belongs to either to $\mathcal{R}_{\pm\alpha^0}$ or $\partial^s Y_{\beta_i^{(l)}}$ with $l + i = j$ and $\Delta_1^i \star \cdots \star \Delta_l^i \sim \Delta_{i+1} \star \cdots \star \Delta_j$.*

The following definition is useful for the proof.

Definition 6.18. Given an atomic piece Δ and a puzzle piece α. We write $\Delta \setminus \alpha$ if $\Delta \sim \Delta_1$ and $\Delta_1 \star \cdots \Delta_k \star s$ the atomic decomposition of α.

Proof. We proceed by induction on $j \geq 1$. The step $j = 1$ is clear. Let us assume the proposition for $j \geq 1$. If both ends of γ_j belong to $\mathcal{R}_{\pm\alpha^0}$, we are back to step $j = 1$. Otherwise, one of the end of γ_j belongs to a certain $\partial^s Y_{\beta_i^{(l)}}$ with $i + l = j$.

We notice that $\beta_i^{(l)}$ belongs to $\mathcal{Y}(\mathbb{S}^{c_{n_i} \star \Delta_1^i \star \cdots \star \Delta_l^i})$. As β_i is simple or satisfies $n_{\beta_i} \leq \mu n_{c_{n_i}}$, we have $n_{\Delta_{l+1}^i} \leq \mu n_{c_{n_i}}$. Also:

$$(c_{n_i} \star \Delta_1^i \star \cdots \star \Delta_l^i) \wedge (\Delta_i \star \cdots \star \Delta_j) \geq n_{c_{n_i}}.$$

Thus the piece Δ_{l+1}^i is equivalent to an atomic piece of $S^{t\Delta_1 \star \cdots \star \Delta_j}$.

Consequently, we can distinguish two cases:

If $\Delta_{j+1} \setminus \beta_i^{(l)}$, then the end corresponding to $\beta_i^{(l)}$ of γ_j is sent to an end of γ_{j+1} in $\partial^s Y_{\beta_i^{(l+1)}}$.

If Δ_{j+1} does not left-divide $\beta_i^{(l)}$ then the end corresponding to $\beta_i^{(l)}$ of γ_j is sent to an end of γ_{j+1} in $\mathcal{R}_{\pm\alpha^0}$ or $\partial^s \beta_{j+1}$. $\square$

By the above description, if one of the ends of γ_j is in $\mathcal{R}_{\pm\gamma^0}$, then γ_j contains a simple piece $S_\pm^2$ of order 2. The associated slice of $f^{n_\beta}(S'_\beta)$ to S is merely γ_j. The length of γ_j is in this case greater than $1/K$.

If none of the ends of γ_j is in $\mathcal{R}_{\pm\alpha^0}$, then both ends of γ_j belong to some $\partial^s Y_{\beta_i^{(l)}}$. By the critical position, there is a hook of $f^{n_\beta}(S')$ in $Y_{\beta_i^{(l)}}$. This hook is in a $\left[\frac{i+M}{20}\right]$-critical position with respect to $\beta_i^{(l)}$. We associate then to S the union of γ_j with the slices of these hooks between γ_j and $\partial^s Y_{\beta_i^{(l)} \star \bigstar_1^{i'} s_-^2}$ (resp. $\partial^s Y_{\beta_i^{(l)} \star s_+^2 \star \bigstar_1^{i'} s_-^2}$) with $i' = \left[\frac{i+M}{20}\right]$. This slice have length greater than:

$$e^{-2c^+\left[\frac{i+M}{20}\right]} e^{-2c^+ n_{\beta_i^{(l)}}} \le e^{-2c^+\left[\frac{i+M}{20}\right]} e^{-2c^+ \max(\mu c_{n_i}, M)} \le e^{-3n_{\Delta_1 \star \cdots \star \Delta_j} c^+}.$$

The following proposition is a consequence of the above discussion:

Proposition 6.19. *For $i < k$, to each curve $S' \in \Sigma_i$ and each product of atomic pieces β such that $S = S'^\beta$ belongs to $\tilde\Sigma_k$, there is a (combinatorial defined) slice of $f^{n_\beta}(S'_\beta)$ only associated to S'^β. This slice is of length greater than $e^{-3n_\beta c^+}$.*

6.3 Strongly regular dynamics are regular

6.3.1 Compactification

We notice that each T_k and $\tilde{T}_k$ are compact for every k. However $T' := \cup_{k \ge 0} T_k$ is not compact, but precompact. Thus to compactify $\cup_{k \ge 0} T_k$ we only need to complete it. This is done canonically by considering the space T of Cauchy sequences quotiented by the relation $(t_i)_i \approx (t'_i)_i$ iff $(d(t_i, t'_i))_i$ converges to zero. Then the distance on T is the limit of the distances between the corresponding terms of the sequences. As the term of a Cauchy sequence of T' have eventually the same first elements, an element of T can be written as an infinite presequence of pieces in $\cup_k \mathcal{Y}_k$ up to $\sim$.

By Proposition 6.15, the map $t \in T_k \mapsto S^t$ is 1-Lipschitz for the C^1-topology. Consequently to an element $t \in T$ corresponds a unique flat and stretched curve S^t. We put $\Sigma := \{S^t\}_{t \in T}$.

We recall that the puzzle pieces of a curve $S \in \Sigma'$ depend only on the geometric position of S and of its derivates $S^{\Delta_1 \cdots \Delta_j}$ with respect to some common pieces. For every $n \ge 0$, for curves $S', S \in \Sigma'$ close enough, there exists an embedding of S onto S' close to the canonical inclusion which map the puzzle pieces of order less than n of S onto those of S'.

Therefore, when we have a Cauchy sequence, we can define the puzzle pieces of order less than n on its limit. This is how the set of puzzle pieces $\mathcal{Y}$ is defined on Σ.

We notice that $(\Sigma, \mathcal{Y})$ is a puzzle algebra, since all the regular properties of curves are closed for the C^1-topology. Moreover it satisfies the third property of the regular dynamics, with $A = \cup_{n \ge 0, S \in \Sigma} f^n(S)$, the first property is obviously satisfied. Also we recall that $(\Sigma_1, \mathcal{Y}_1)$ is included in $(\Sigma, \mathcal{Y})$.

6.4 Second property of the regular dynamics

The following proposition is the remaining property implying that strongly regular dynamics are regular. Furthermore this is the key proposition to show the abundance of strongly regular dynamics.

Proposition 6.20. *Let $k \geq 1$ and let f be k-strongly regular. Let S be a curve of Σ_i, $i \leq k$. Let $\mathcal{E}_k^j(S)$ be the set of points of S that do not belong to a piece of $\mathcal{Y}_k(S)$ of order less than $j \leq k$.*

Let $S = S'^\beta$ be a curve of $\tilde{\Sigma}_i$, $i \leq k$ and $S' \in \Sigma_{i-1}$ and $\beta = \Delta_1 \star \cdots \star \Delta_n$. Let $\mathcal{E}_k^j(S)$ be the set of points of $f^{n_\beta}(S'_\beta)$ that do not belong to a piece of $\mathcal{Y}_k(S)$ of order less than $j \leq k$.

In both cases, its Lebesgue measure satisfies the following estimate:

$$leb(\mathcal{E}_k^j(S)) \leq K e^{-\frac{c}{4}\min(j,k-i)}.$$

Moreover the connected components of $\mathcal{E}_k^j(S)$ have diameter greater than $e^{-2(j+M)c^+}$.

Proof. Let us prove this proposition by induction k. If i belongs to $\{k, k-1\}$, $\mathcal{Y}_k(S)$ consists of simple pieces and the propsiiton is obvious. Let $i < k - 1$.

For every $S \in \Sigma_i$, let $l = k - 2 - i$. For $m \leq l$ let $\Delta_{m\pm} := (c_m(S), c_{m+1}(S))_\pm$. For $S \in \Sigma_i$, the curve $S^{\Delta_{m\pm}}$ belongs to $\tilde{\Sigma}_{i+m+2}$, by Remark 5.5. It is also the case if $S = S^\beta$ belongs to $\tilde{\Sigma}_i$ and $S_{\Delta_{m\pm}}$ intersect $f^{n_\beta}(S'_\beta)$. If the intersection is empty, we put $\mathcal{E}_{k-1}^{j-n_{\Delta_m}}(S^{\Delta_{m\delta}}) = \emptyset$.

By the h-property atomic pieces, we have:

$$leb(\mathcal{E}_k^j(S)) \leq \sum_{m=0,\ \delta \in \{+,-\}}^{l} e^{-\frac{c}{3}n_{\Delta_m}} leb(\mathcal{E}_{k-1}^{j-n_{\Delta_m}}(S^{\Delta_{m\delta}})) + leb(S_{\sigma_{l+1}}),$$

with $\Delta_m = (c_m, c_{m+1})$.

$$\Rightarrow leb(\mathcal{E}_k^j(S)) \leq \sum_{m=0,\ \delta \in \{+,-\}}^{l} K e^{-\frac{c}{3}n_{\Delta_m}} e^{-\frac{c}{4}\min(j-n_{\Delta_m}, k-i-m-2)} + leb(S_{\sigma_{l+1}}),$$

$$\Rightarrow leb(\mathcal{E}_k^j(S)) \leq \sum_{m=0}^{l} 2K \max\left(e^{-\frac{c}{12}n_{\Delta_m}} e^{-j\frac{c}{4}}, e^{-\frac{c}{3}n_{\Delta_m}+\frac{c}{4}(m+2)} e^{-(k-i)\frac{c}{4}}\right) + leb(S_{\sigma_{l+1}}).$$

As $n_{\Delta_m} \geq M$, we get:

$$leb(\mathcal{E}_k^j(S)) \leq \frac{1}{2} e^{-\frac{c}{4}\min(j,k-i)} + leb(S_{\sigma_{l+1}}).$$

Also $leb(S \cap Y_{\sigma_{l+1}})$ is smaller than the square root of the product of e^{-2cM} with the width of $Y_{c_{l+1}}$ which is smaller than $e^{-c^- n_{c_{l+1}}}$.

$$\Rightarrow leb(S_{\sigma_{l+1}}) \leq e^{-cM} e^{-c^- n_{c_{l+1}}+1} \leq e^{-c^- n_{\sigma_l}} \leq \frac{1}{2} e^{-\frac{c}{4}\min(j,k-i)}.$$

This finishes the proof of the first statement.

Let us upper bound the diameter of the connected component of $\mathcal{E}_k^j(S)$. Such components contain $S^{\Delta_1 \star \cdots \star \Delta_l \star \alpha}$ with α simple or of the form σ_m. Also we have $\sum n_{\Delta_i} + n_\alpha \leq j + M$. We have:

$$|S_{\Delta_1 \star \cdots \star \Delta_l \star \alpha}| \geq e^{-2c^+ \sum n_{\Delta_i}} |S_\alpha|.$$

If α is simple, the width $|S_\alpha|$ is greater than $e^{-c^+ n_\alpha}$ and so the bad component has width greater than $e^{-2(j+M)c^+}$.

Otherwise $\alpha = \sigma_m$. By the $m' = \left[\frac{M+m}{20}\right]$-critical position, the length of S_{σ_m} is greater than:

$$\frac{1}{K}e^{cM}e^{-2c^+(M+1)}|\mathbb{S}_{c_m \star c_+^2 \star \bigstar_1^{m'} s_-^2}|$$

$$\geq \frac{1}{K}e^{cM-2c^+M}e^{-(n_{c_m}+1+2m')c^+} \geq \frac{1}{K}e^{(c-c^+)M}e^{-n_{\sigma_m}c^+-2\left[\frac{M+m}{20}\right]c^+}.$$

Consequently,

$$|S_{\Delta_1 \star \cdots \star \Delta_l \star \sigma_m}| \geq \frac{1}{K}e^{-2c^+ \sum_i n_{\Delta_i}+n_{\sigma_m}c^+-2\left[\frac{M+m}{20}\right]c^++(c-c^+)M} \geq \frac{1}{K}e^{-2c^+(j+M)}.$$

$\square$

6.5 Basin and uniqueness of the SRB measure

Given an f-invariant, Borelian probability λ, the *basin* of λ is the set $B(\lambda)$ of the points $x \in \mathbb{R}^2$ such that the average Dirac measures along the orbit of x converges to λ in the weak sense:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = \int \phi d\lambda \quad \text{for any continuous function } \phi.$$

The measure λ is *physical* if the basin $B(\lambda)$ has a positive Lebesgue measure.

Let us now show that the measure ν constructed in 3.11 is unique, physical and that its basin contains the common stable leaves W_c^s for every common sequence of regular pieces in $\mathcal{Y}$ for every curve $S \in \Sigma$.

We begin by proving the following lemma:

Lemma 6.21. *For every $S \in \Sigma$, the set:*

$$B(S) := \bigcup \{W_c^s : c \text{ common sequence in } \mathcal{Y} \text{ from } S\}$$

has relative measure in Y_e greater than $Ke^{\frac{-cM}{6}}$.

The proof is very similar and simpler than the parameters selection and so worth to be read.

Proof. A point $z \in Y_e$ is *k-regular* with respect to $S \in \Sigma$ if there exists a common sequence $(\alpha_l)_{l=0}^k$ from S such that z belongs to the support of $c_k := \bigstar_{l=0}^k \alpha_l$ of depth k.

The point z is *regular* if its k-regular for every $k \geq 0$. Let us put $W_{loc}^S(z) := \bigcap_{k \geq 0} Y_{(c_k)}$.

Let $y_0 \in \left[-2\theta, 2\theta\right]$ and let Δ be the intersection of the line $\{y = y_0\}$ with Y_e.

For every k, the intersection of $Y_{(c_k)}$ with Δ is a segment whose tangent space is in χ. Moreover by the h-times property, this segment is exponentially small with k. Thus the intersection of $W_{loc}^s(x)$ with Δ consists of a single point. Also for every $i \leq k$ the curves $\partial^s Y_{(c_k)}$ is $(Kb)^{n_{c_i}}$-close to plaques of $\mathcal{F}_{n_{c_i}}$. Thus $W_{loc}^S(x)$ must be a C^1-curves with tangent space $b^{1/2}$-exponentially fast contracted. As its normal space is exponentially fast expanded, its is a local stable manifold.

Let us now estimate the measure of k-regular points in Δ, with respect to S.

The following estimate is very similar to the parameter selection estimate in [Yoc]. Hence we almost write words for words it. We put $\bar{\theta} := c/20$.

We first define on Δ a sequence of integer valued "random" variables:

If $z \in D$ is not k-regular, we define $X_k(z) := 0$.

Otherwise we put $c_{k-1} := \bigstar_{j=0}^{k-1}\alpha_j$. If $f^{n_{c_k}-1}(z)$ belongs to the support of a maximal piece $\alpha_k \in \mathcal{Y}(S_{c_{k-1}})$, we put $X_k := n_{\alpha_k}$. Otherwise we put $X_k := \max(M, \mu n_{c_k}) + 2$.

We put $N_k := \sum_{j \leq k} X_j$. we observe that if z is k-regular but not $k+1$-regular, then:

$$\sum_{\substack{l=1 \\ X_l \geq M}}^{k+1} X_l > \mu N_k \geq \mu(k+1).$$

By Property 2 of regular maps and $Dist$-property of regular pieces, we have the conditional probability:

$$\mathbb{P}(X_1, \ldots, X_{k-1}|X_k > m) \leq Ke^{-\frac{c}{4}m}$$

valid for all $k \geq 1$, $m \geq 0$ in view of the definition of X_k. We now set:

$$\tilde{X}_k = \begin{cases} 0 & \text{if } X_k < M \\ X_k & \text{if } X_k \geq M \end{cases}$$

$$\tilde{N}_k = \sum_1^k \tilde{X}_l.$$

Let us estimate $\{\tilde{N}_k \geq \mu k\}$. The following is a standard calculation in large deviation theory. We will estimate:

$$\int_\Delta e^{\bar{\theta}\tilde{N}_k(z)} dz$$

We have

$$e^{\bar{\theta}\tilde{N}_k(z)} = e^{\bar{\theta}\tilde{N}_{k-1}(z)}e^{\bar{\theta}\tilde{X}_k(z)}.$$

On one side we have $X_k(z) = \tilde{X}_k(z) = 0$ if z is not $k-1$-regular. On the other side, if Δ_k is an interval of Δ on which $(X_i)_{i<k}$ is constructed, we have

$$\int_{\Delta_k} e^{\bar{\theta}\tilde{X}_k(z)} dz = \sum_{m \geq 0} e^{\bar{\theta}m} \int_{\Delta_k} \mathbb{1}_{\{\tilde{X}_k=m\}} dz$$

$$\leq (1 + K \sum_{m \geq M} e^{(\bar{\theta}-c/4)m})|\Delta_k| \leq (1 + Ke^{(\bar{\theta}-c/4)M})|\Delta_k|.$$

We conclude that

$$\int_\Delta e^{\bar{\theta}\tilde{N}_k(z)} dz \leq (1 + Ke^{(\bar{\theta}-c/4)M}) \int_\Delta e^{\bar{\theta}\tilde{N}_{k-1}(z)} dx \leq (1 + Ke^{(\bar{\theta}-c/4)M})^k|\Delta|.$$

It follows that:

$$\frac{1}{|\Delta|}m\{\tilde{N}_k \geq \mu k\} \leq \frac{[1 + Ke^{(\bar{\theta}-c/4)M}]^k - 1}{e^{\bar{\theta}\mu k} - 1} =: u_k,$$

as $\tilde{N}_k \geq 0$ everywhere. Consequently the relative measure of R in Y_e greater than $1 - \sum_{k \geq 0} u_k$. With $\bar{\theta} = c/20$, one shows that:

$$\sum_{k \geq 0} u_k \leq K e^{\frac{-c}{6}}$$

$\square$

Actually we can show that $B(S)$ and so its stable leaves do not depends on $S \in \Sigma$. This will be clear after Section 7.2. However we do not need to show such fact and so we put $B = \cup_{S \in \Sigma} B(S)$. The stable manifolds of B form the leaves of a lamination $\mathcal{L}$ on B. Actually all these leaves are of the form W_c^s for some common sequences of pieces c. The definition of lamination suppose usually the continuity of the tangent space to the leaves. This is a consequence of the following lemma.

Lemma 6.22. *The tangent space of the leaves $\mathcal{L}$ is a 2-Lipschitz function of the point $z \in B$.*

Proof. Let $z, z' \in B$. We want to show that that the angle between $T_z W^s$ and $T_{z'} W^s$ is less than twice the distance d between z and z'. We remind that z and z' satisfies the projective Collet-Eckmann condition $\mathcal{PCE}^\infty$ since they belongs to some common stable leaves. This implies that $\angle(T_z W^s, e_n(z)) \leq (K e^{2Mc^+} b)^n$ and $\angle(T_{z'} W^s, e_n(z')) \leq (K e^{2Mc^+} b)^n$ for every $n \geq 0$, by the first statement of Proposition 4.4.

By binding, every point $z'' \in [z, z']$ satisfies $\mathcal{PCE}^n$, if the distance d between z and z' is less than $e^{-2c^+ - (\lambda + c^+)n}$. By the last statement of Corollary 4.5, for every $z'' \in [z, z']$ we have:

$$\|d_z \angle(e_n(z''), e_1(z))\| \leq \sqrt{b}.$$

As $\|d_z e_1\| = 1$, we have:

$$\|\angle(T_z W^s, T_{z'} W^s)\| \leq |\angle(T_z W^s, e_n(z))| + |\angle(e_n(z), e_n(z'))| + |\angle(e_n(z'), T_{z'} W^s)|$$

$$\leq (K e^{2Mc^+} b)^n + (1 + \sqrt{b})d \leq (K e^{2Mc^+} b)^{-\frac{\log d + 2c^+}{\lambda + 2c^+} - 1} + (1 + \sqrt{b})d \leq 2d$$

$\square$

We recall that Σ is the family of leaves of the lamination $\tilde{X}$.

The physicality of the SRB measure ν and its uniqueness follows from the following lemma:

Lemma 6.23 (Lemma 4.11 [Bv06])**.** *Let ν be any R-invariant probability measure absolutely continuous along the unstable leaves. Then ν is ergodic for the return map R and its basin:*

$$B(\nu) = \{z \in \tilde{X} : \lim \frac{1}{n} \sum_{j=0}^{n-1} \delta_{R^k(z)} = \nu\}$$

intersects every leaf $S^t \in \Sigma$ on a full measure subset.

Proof. We adapt to our setting the proof of [Bv06]. The main step is the following Sublemma. A set $F \subset \tilde{X}$ is called $\mathcal{L}$-*compatible* if for every $z \in B \cap F$, $W^s(z) \cap \tilde{X}$ is included in F.

For $S^t \in \Sigma$, let us denote by m_t the Lebesgue measure on S^t.

Sublemma 6.24. *Let $F \subset \tilde{X}$ be an R-invariant $\mathcal{L}$-compatible set such that $m_t(S^t \cap F) > 0$ for some $S^t \in \Sigma$. Then $m_{t'}(S^{t'} \setminus F) = 0$ for every $S^{t'} \in \Sigma$.*

Proof. We recall that almost every point of $z \in S^t$ can be written as the single element of $\cap_{i \geq 1} S_{\alpha_1 \star \cdots \star \alpha_i}$, with $(\alpha_i)_i \in \mathcal{Y}^{\mathbb{N}}$.

Thus for almost every point of density $z \in S^t \cap F$ satisfies the previous property.

By the Distortion property, we have:

$$\frac{m_{t \cdot \alpha_1 \cdots \alpha_i}(S^{t \cdot \alpha_1 \cdots \alpha_i} \setminus F)}{m_{t \cdot \alpha_1 \cdots \alpha_i}(S^{t \cdot \alpha_1 \cdots \alpha_i})} \leq K \frac{m_t(S_{t \cdot \alpha_1 \cdots \alpha_i} \setminus F)}{m_t(S_{t \cdot \alpha_1 \cdots \alpha_i})} \to 0.$$

$$\Rightarrow \frac{m_{t \cdot \alpha_1 \cdots \alpha_i}(S^{t \cdot \alpha_1 \cdots \alpha_i} \cap F)}{m_{t \cdot \alpha_1 \cdots \alpha_i}(S^{t \cdot \alpha_1 \cdots \alpha_i})} \to 1.$$

Let us suppose for the sake of contradiction that $m_{t'}(S^{t'} \setminus F) \neq 0$.

We remark that the complement of F^c of F is as well $\mathcal{L}$-compatible and so we have also the existence of a sequence $(\alpha'_i)_i$ such that:

$$\frac{m_{t' \cdot \alpha'_1 \cdots \alpha'_i}(S^{t' \cdot \alpha'_1 \cdots \alpha'_i} \cap F)}{m_{t' \cdot \alpha'_1 \cdots \alpha'_i}(S^{t' \cdot \alpha'_1 \cdots \alpha'_i})} = \frac{m_{t' \cdot \alpha'_1 \cdots \alpha'_i}(S^{t' \cdot \alpha'_1 \cdots \alpha'_i} \setminus F^c)}{m_{t' \cdot \alpha'_1 \cdots \alpha'_i}(S^{t' \cdot \alpha'_1 \cdots \alpha'_i})} \leq K \frac{m_{t'}(S^{t'}_{\alpha'_1 \star \cdots \star \alpha'_i} \setminus F^c)}{m_{t'}(S^{t'}_{\alpha'_1 \star \cdots \star \alpha'_i})} \to 0.$$

Thus, for every $\eta > 0$ we have the existence of two curves $S, S' \in \Sigma$ such that $\frac{m(S \cap F)}{m(S)} > 1 - \eta$ and $\frac{m(S' \cap F)}{m(S')} < \eta$. This contradicts the fact that $\frac{m(S \cap F)}{m(S)} < K \frac{m(S' \cap F)}{m(S')}$ by Lipschitz holonomy along $\mathcal{L}$. $\qquad\square$

In order to prove Lemma 6.23, let $\phi : \tilde{X} \to \mathbb{R}$ be a continuous function and $\eta > 0$ a real number. Let E be the set of points z such that the forward-time average of ϕ on the R-orbit of z converges and is less than η. Then E is R-invariant and it is $\mathcal{L}$-compatible because the leaves of $\mathcal{L}$ are stable manifolds. Suppose that $\nu(E) > 0$. As ν is absolutely continuous along the leaves, we must have $m_t(S^t \cap E) > 0$ for some $S^t \in \Sigma$. Using the absolute continuity once more, we get that E has full ν-measure: $\nu(\tilde{X} \setminus E) = 0$. This shows that time-averages of continuous functions are constant ν-almost everywhere. Therefore, ν is ergodic.

The proof of the last statement of the lemma is similar. By ergodicity, the basin of ν has full ν-measure, and so it intersects some long unstable leaf on a positive arc-length measure set. By the sublemma, the intersection indeed has full measure, for every long unstable leaf. $\qquad\square$

The next proposition is an easy consequence of all the previous works:

Proposition 6.25. *A strongly regular dynamics $(\hat{A}, \Sigma, \mathcal{Y})$ has exactly one $\hat{f}$-invariant probability measure along unstable leaves. Moreover, ν is ergodic, its density is bounded, its basin contains every common stable manifold and every unstable manifold of A.*

Proof. The proposition is a mere consequence of the work done if we replace $\hat{f}$ by R and $\hat{A}$ by $\tilde{X}$. To get the proposition, we remark that $\hat{A} = \cup_{n \in \mathbb{Z}} \hat{f}^n(\tilde{X})/ \sim$, and that R can be identified to the first return map to $\tilde{X} \approx \tilde{X} \times \{0\} \subset \hat{A}$ of $\hat{f}$. $\qquad\square$

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Part III

Parameters Selection

7   Combinatorial classes and finite subsets $\check{\Sigma}_k$

7.1   Definitions

Let f_0 be a k-strongly regular dynamics. Let $\Sigma_k(0)$ and $\mathcal{Y}_k(0)$ be the set of stretched flat curves and the set of puzzle pieces given by the definition of the k-strong regularity. A homotopy $(f_t)_{t\in[0,1]}$, starting at f_0 in the space of the C^2 dynamics is a k-*combinatorial equivalence* if:

- for every $t \in [0,1]$, the dynamics f_t is k-strongly regular,

- for each $S \in \Sigma_k(0)$, there exists a homotopy $(S(t))_{t\in[0,1]}$ such that $S(0) = S$ and for $t \in [0,1]$ the set $\Sigma_k(t)$ of stretched, flat curves of f_t is equal to $\{S(t);\ S \in \Sigma_k(0)\}$,

- for each $\alpha \in \mathcal{Y}_k(0)$ there exists a homotopy $(S_\alpha(t))_{t\in[0,1]}$ such that $S(0) = S$ and for $t \in [0,1]$ the set of puzzle pieces $\mathcal{Y}_k(t)$ of f_t is

$$\Big\{\alpha(t) := \{S_\alpha(t), n_\alpha\};\quad \alpha \in \mathcal{Y}_k(0)\Big\},$$

- for every $t \in [0,1]$, for every $S \in \Sigma_k(0)$, every $j \le k$,

$$c_j(S)(t) = c_j\big(S(t)\big).$$

Two k-strongly regular dynamics are k-*equivalent* if they be linked by a k-combinatorial equivalence.

We notice that every dynamics in a same k-equivalent class have the same first return M. However, the perturbation B is not necessarily the same along the homotopy. In order to prove the main theorem, let us consider the special case of a perturbation B constant along the homotopy.

For M fixed and then B small, a k-*combinatorial component* is a maximal interval w of $\mathbb{R}$ such that $(f_{a,B})_{a\in w}$ is a k-combinatorial equivalence.

7.2   Finite subsets $\check{\Sigma}_k$

In order to do the parameters selection, we shall deal with a θ^k-dense subset of curves $\check{\Sigma}_k$ of curves in Σ_k, with low cardinality. By θ^k-density, if these curves are in good position with some common pieces with some room, this implies that all the curves of Σ_k are in good position with the same common pieces. To make work the parameters selection, we express this subset $\check{\Sigma}_k$ combinatorially and prove that its cardinality is less than an exponential function of k.

We recall that for a k-strongly regular map f, the spaces Σ_k and $\Sigma_k \cup \tilde{\Sigma}_k$ are parametrized by Cantor sets T_k and $T_k \cup \tilde{T}_k$:

$$\Sigma_k = \{S^t;\ t \in T_k\}\quad\text{and}\quad \tilde{\Sigma}_k = \{S^t;\ t \in \tilde{T}_k\}.$$

The homotopies of the space $\Sigma_k \cup \tilde{\Sigma}_k$ induce a homeomorphism between the transversal space $T_k \cup \tilde{T}_k(a)$ and $T_k \cup \tilde{T}_k(a')$ for a and a' in a same k-component. The homotopies of $\mathcal{Y}_k$ induce also a homeomorphism on the space of finite presequences $\mathcal{D}_k(a), \tilde{\mathcal{D}}_k(a)$ with $\mathcal{D}_k(a'), \tilde{\mathcal{D}}_k(a')$. This homeomorphism pass to the quotient by $\sim$.

Let G_k be the quotient space: $(\mathcal{D}_k \cup \tilde{\mathcal{D}}_k)/\sim$ which depends only on of the k-combinatorial component. We notice that G_k is locally compact, and is equipped with a structure of pseudo-semi-group via the concatenation operation $\cdot$.

In order to express combinatorially $\check{\Sigma}_k$, we shall associate to each $d \in G_k$ a sequence of divisors τ_d which depends only on the k-combinatorial component of f.

Proposition 7.1. *For every $d \in G_k$, there exists a family $\tau_d^k := \{d_i\}_{i=0}^{t_d}$ of element in $G_k/\sim$ satisfying the following properties:*

(i) $d/d_{t_d}/d_{t_{d-1}}/\cdots/d_1/d_0$, with d_0 simple or of the forms $(e, e)_\pm$ or $(e, s)_\pm$ with s simple,

(ii) $n_{d_i} < n_{d_{i+1}} \leq 2M \cdot n_{d_i}$ and $n_d \leq 2M \cdot n_{t_d}$,

(iii) $\mathbb{S}^{d_i}$ is well defined and belongs to Σ_k, with $\mathbb{S} = S^{(s_-^2)_{i \leq 0}}$,

(iv) τ_d^k is equal to τ_d^j for $j \leq k$ and $d \in G_j$,

(v) if $d \in \mathcal{D}_k/\sim$ corresponds to a common pieces formed by puzzle pieces in $\mathcal{Y}_k$, then $d_{t_d} = d$.

By the last property we write τ_d instead of τ_d^k.

The family $(\tau_d)_d$ share some analycal properties with the concept of favorable times of [BC91].

Proof. We prove this proposition by induction on $k \geq 1$.
Step $k = 1$.

The semi-group G_1 consist of finite product of simple pieces. Let $d := (s_j)_{j=-n}^0$ with $(s_j)_j$ simple pieces. We put $d_i := (s_j)_{j=i}^0$ and $\tau_d := \{d_i; \quad i \in [\![-n, 0]\!]\}$.
Step $k \to k+1$. For $d \in \mathcal{D}_k \cup \tilde{\mathcal{D}}_k$, we write $\tau_d := \tau_{d'}$ if $d' \in G_k$ is the $\sim$ class of d.

Let $d \in \tilde{\mathcal{D}}_{k+1}$. A representent of d is of the form $d' \cdot \Delta$ with $d' \in \mathcal{D}_k \cup \tilde{\mathcal{D}}_k$ possibly empty, and Δ of the form $(c_i(S^t), c_{i+1}(S^t))_\pm$ where S^t is in $\Sigma_k \cup \tilde{\Sigma}_k$ with t/d' and $i \leq k-1$. By remark 5.5, $c_i(S^t)$ belongs to G_k. We put:

$$\tau_d := \tau_{c_j(S^t)} \cup \left\{ d_l' \cdot \Delta : \ d_l' \in \tau_{d'} \text{ s.t. } n_{d_l^1} \geq \mu n_\Delta \right\}, \quad \text{if } c_i \neq e.$$

Otherwise, $c_i = e$ and we put:

$$\tau_d := \{\Delta\} \cup \{d_l' \cdot \Delta : \ d_l' \in \tau_{d'}\}.$$

Property (i) is clear.

Let us prove Property (iii). For $d_l' \in \tau_{d'}$, by induction $\mathbb{S}^{d_l'}$ is well defined and belongs to $\Sigma_k \cup \tilde{\Sigma}_k$. By Proposition 6.15,

$$d(\mathbb{S}^{d_l'}, S^t) \leq b^{\frac{n_{d_l'}}{8}}.$$

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As $n_{d'_l} \geq \mu n_\Delta$, the curves $\mathbb{S}^{d'_l}$ and S^t are $b^{\mu n_{c_i}/8}$-close for the C^1-topology. Thus $\big(c_i(S^t), c_{i+1}(S^t)\big) = \big(c_i(\mathbb{S}^{d'_l}), c_{i+1}(\mathbb{S}^{d'_l})\big)$. Consequently $\big(c_i(S^t), c_{i+1}(S^t)\big)_\pm = \big(c_i(\mathbb{S}^{d'_l}), c_{i+1}(\mathbb{S}^{d'_l})\big)_\pm$ and the curve $\mathbb{S}^{d'_l \cdot \Delta}$ is well defined and belongs to Σ_{k+1}, since $\mathbb{S}^{d'_l}$ belongs to Σ_k and $i \leq k-1$.

Let us prove the first inequality of (ii): $n_{d_{j+1}} \leq 2M \cdot n_{d_j}$ for every $j \in \{0, \ldots, t_d - 1\}$.

This inequality is obviously a consequence of the induction hypothesis when d_j and d_{j+1} belong both to τ_Δ or both to $\tau_{d'} \cdot \Delta$. This inequality is also obvious if $c_i = e$. Let us suppose that $c_i \neq e$.

When d_j belongs to τ_Δ but d_{j+1} belongs to $\tau_{d'} \cdot \Delta$, by (v) we have $d_j = c_j$ and $d_{j+1} = d'_l \cdot \Delta$ with $n_{d'_l} \leq 2M\mu n_\Delta$. Thus n_{d_j} is less than:

$$2M\mu n_\Delta + n_{c_j} + M + 1 \leq (1 + 2M\mu)n_{c_j} + (1 + 2M\mu)(M+1) \leq 2Mn_{c_j} = 2Mn_{d_j},$$

since $n_{c_j} \geq 2$ and M is large.

The last inequality of (ii) will be proved below, after the definition of τ on $\mathcal{D}_{k+1}$.

Let $\underline{d \in \mathcal{D}_{k+1}}$. If d belongs to $\mathcal{D}_k$, then τ_d is given by induction. Otherwise d is of the form $d' \cdot \alpha \cdot s$, with $\alpha \in \mathcal{Y}_k(S^t)$ s.t. $d' \in \tilde{\mathcal{D}}_k$ divides $t \in \tilde{T}_k$ and with s is a product of simple pieces. As $\alpha \cdot s$ belongs to $\mathcal{D}_k$, we can put:

$$\tau_d := \tau_{\alpha \cdot s} \cup \big\{d'_i \cdot \alpha \cdot s : \ d'_i \in \tau_{d'}, \ n_{d'_i} \geq \mu n_\alpha \big\}.$$

In the above expression, α is identified to its class in $G_k \subset G_{k+1}$. Actually this splitting is not unique, and so we suppose that n_α is minimal.

Properties (i), (iv) and (v) are now clear.

Property (iii) follows from the induction for $d_i \in \tau_{\alpha \cdot s}$. For $d_i \in \tau_d \setminus \tau_{\alpha \cdot s}$, we have $d_i = d'_i \cdot \alpha \cdot s$. As d'_i divides t, the curves $\mathbb{S}^{d'_i}$ and S^t are $b^{n_{d'_i}/8}$-C^1-close.

As $n_{d'_i} \geq \mu n_\alpha$, by using the atomic decomposition, one sees that the piece α is equivalent to a piece $\alpha' \in \mathcal{Y}_k(\mathbb{S}^{d'_i})$. Thus $\mathbb{S}^{d'_i \cdot \alpha \cdot s}$ is well defined and belongs to Σ_k.

The first inequality of (ii) is proved similarly as for $d \in \tilde{\mathcal{D}}_{k+1}$. Let $i \geq 0$ be s.t. $d_i \in \tau_{\alpha \cdot s}$ is maximal and let $d'_j \in \tau_{d'}$ s.t. $d_{i+1} = d'_j \cdot \alpha \cdot s$. We have $\mu n_{\alpha \cdot s} \leq n_{d'_j} \leq 2\mu M n_{\alpha \cdot s}$. We will prove that for a puzzle piece $\alpha \in \mathcal{Y}_k$, the order of the maximal divisor of τ_α is greater than $\frac{2}{3M} n_\alpha$. This implies that $n_{d_i} \geq \frac{2}{3M} n_{\alpha \cdot s}$. Thus:

$$\frac{n_{d_{i+1}}}{2M} = \frac{n_{d'_j} + n_{\alpha \cdot s}}{2M} \leq \frac{2\mu M n_{\alpha \cdot s}}{2M} + \frac{3M n_{d_i}}{4M} \leq n_{d_i}.$$

$\underline{\text{Second inequality of } (ii)}$. We notice that when Δ is an atomic pieces or a simple piece, the maximal $d \in \tau_\Delta$ satisfies:

$$\frac{n_d}{n_\Delta} \geq \frac{1}{M}.$$

Let α be a puzzle piece of $\mathcal{Y}_k$. Let us develop it in its atomic decomposition: $\alpha = \Delta_n \star \cdots \star \Delta_1 \star s$. As $\alpha \sim \Delta_n \cdots \Delta_1 \cdot s$, we have $\tau_\alpha = \tau_{\Delta_n \cdots \cdot \Delta_1 \cdot s}$. Let $i \in \{1, \ldots, n\}$ be such that the maximal $d \in \tau_\alpha$ belongs to $\tau_{\Delta_i} \cdot \Delta_{i-1} \cdots \Delta_1 \cdot s$. We have for every $j > i$ the existence of $m(j) \in \{i, \cdots, j-1\}$ minimal such that:

$$\mu n_{\Delta_{m(j)}} \geq \frac{1}{M} n_{\Delta_j} + n_{\Delta_{j-1}} + \cdots + n_{\Delta_{m(j)+1}}.$$

We notice that the preimage of $m^{-1}(\{k\})$ is an interval of $\{i+1, \ldots, n\}$. We regroup $\Delta_n \cdots \Delta_i$ following these intervals: $\Delta'_j \cdots \Delta'_i$ with $\Delta'_i = \Delta_i$ and Δ'_l equal to the concatenation of the Δ_p belonging to some interval $m^{-1}(\{k\})$. We notice that we have:

$$M\mu n_{\Delta'_l} \geq n_{\Delta'_{l+1}}.$$

And so:

$$n_{\Delta_i} \geq \frac{2}{3} \sum_{j=i}^{j} n_{\Delta_j}.$$

Therefore,

$$n_d \geq \frac{1}{M} n_{\Delta_i} \geq \frac{2}{3M} n_\alpha.$$

Thus we get that when Δ is an atomic piece or a puzzle piece of $\mathcal{Y}_k$, the maximal $d \in \tau_\Delta$ satisfies:

$$\frac{n_d}{n_\Delta} \geq \frac{2}{3M}.$$

Proceeding exactly as we just did, we get the last inequality of (ii). $\qquad\square$

We put now $\check{\mathcal{D}}_k := \{d \in G_k : n_d \leq \mu k \text{ and } \mathbb{S}^d \text{ is well define}\}$. Let $\check{\Sigma}_k := \{\mathbb{S}^d : d \in \check{\mathcal{D}}_k\}$. We recall that $\mathbb{S}^{s^2_-} = \mathbb{S}$. We endow $\check{\Sigma}_k$ with the order relation $\mathbb{S}^d \succeq \mathbb{S}^{d'}$ if d/d'.

Proposition 7.2. *The following properties hold:*

(i) *The cardinality $\#_k$ of $\check{\Sigma}_k$ is less than $Ke^{5\mu kc}$,*

(ii) *For every $S \in \Sigma_k \cup \tilde{\Sigma}_k$, the exists $S' \in \check{\Sigma}_k$ such that $b^{\frac{\mu k}{16M}}$-C^1-close to S',*

(iii) *$\check{\Sigma}_k$ is included in $\Sigma_{[1+\mu k]}$, with $[x]$ the integer part of $x \in \mathbb{R}$.*

Proof. (i) As each curves of $\check{\Sigma}_k$ contains a unique combinatorially defined, slice of $f^{[\mu k]}(\mathbb{S})$ of length at least $K^{-1}e^{-3[\mu k]c^+}$ by proposition 6.19, we get that $\# \leq Ke^{5c^+\mu k}$.

(ii) For every $S \in \Sigma_k \cup \tilde{\Sigma}_k$, there exists $d \in \mathcal{D}_k \cup \tilde{\mathcal{D}}_k$ such that $S = S^{td}$ with $S^t \in \Sigma_1$. Let $d' \in \tau_d$ be s.t. $n_{d'} \in \left[\frac{\mu k}{2M}, \mu k\right]$. The C^1-distance between S and $\mathbb{S}^{d'} \in \check{\Sigma}_k$ is less than $b^{d \wedge d'} \leq b^{\frac{\mu k}{16M}}$.

(iii) Let us define an integer valued *degree* function d^o on the set of puzzle pieces, atomic pieces and curves.

For a simple piece α, let $d^o(\alpha) := 1$. For every atomic piece $\Delta = (c_j(S), c_{j+1}(S))_\pm$ with c_j a common pieces of depth j, let $d^o(\Delta) = j + 3$. For every puzzle piece of $\alpha \in \mathcal{Y}_k$, let $d^o(\alpha) = \sum_i d^o(\Delta_i) + d^o(S)$ with $\bigstar_i \Delta_i \star s$ the atomic decomposition of α.

For every curves $S \in \Sigma_j \cup \tilde{\Sigma}_j \setminus (\Sigma_{j-1} \cup \tilde{\Sigma}_{j-1})$ put $d^o(S) = j$, for every $j \leq k$, with $\Sigma_0 = \emptyset$.

We note that $d^o(\Delta) \leq n_\Delta$ for every atomic piece and $d^o(\alpha) \leq n_\alpha$ for every puzzle piece. For $S \in \check{\Sigma}_k$ there exists $d = (\alpha_i)_{i=-n}^0 \in \check{\mathcal{D}}_k$ s.t. $S = \mathbb{S}^d$. Also we have:

$$d^o(S) \leq d^o(\mathbb{S}) + \sum_i d^o(\alpha_i) \leq 1 + n_d \leq 1 + \mu k.$$

$\qquad\square$

8   Motion of the regular puzzle pieces

We are interested in the dependence of any puzzle piece Y_α of order $n_\alpha \leq k$, with respect to a in a k-regular combinatorial component w.

8.1   Weak estimate

Let a_0 be in a k-regular combinatorial component w. In other words $f := f_{a_0, B}$ is k-strongly regular. Let $\alpha \in \mathcal{Y}_k(a_0)$. Let χ' be a unit vector field on $\partial^s Y_{\alpha(a_0)}$ included in $\chi_{\alpha(a_0)}$.

Proposition 8.1. *There exists a neighborhood V_a of a_0 in w, a C^2-function $\rho_\alpha : V_a \times \partial^s Y_{\alpha(a_0)} \to \mathbb{R}^2$, such that for every $z \in \partial^s Y_{\alpha(a_0)}$:*

   (i) $\rho_\alpha(a_0, z) = z$,

   (ii) $\|\partial_a \rho_\alpha(a_0, z)\| \leq K e^{2c^+ n_\alpha}$, *and* $\partial_a \rho_\alpha(a_0, z) \in \chi'$.

   (iii) $\rho_\alpha(a, \partial^s Y_{\alpha(a_0)})$ *is formed by two curves that intersect each curve of $\partial^s Y_{\alpha(a)}$ at a nonempty open subset, for every $a \in w$.*

Proof. Case $n_\alpha \leq M$.

   By LemmaA.1, we have for all $M \geq 2$, $-\frac{1}{2} \leq \partial_a \alpha^m(a) \leq -\frac{1}{3}$. As $(\hat{\alpha}_m(a))^2 = \alpha^m(a) - a$, we have:

$$\partial_a \hat{\alpha}^m(a) = \frac{1}{2} \left[\frac{\partial_a \alpha^m(a)}{\hat{\alpha}^m(a)} - \frac{1}{\hat{\alpha}^m(a)} \right]$$

$$\Rightarrow |\partial_a \hat{\alpha}^m(a)| \geq \frac{1}{2|\hat{\alpha}^m(a)|}.$$

By Lemma A.2 , $|\hat{\alpha}^m(a)| \geq K^{-1} 2^{-m}$. This implies that $|\partial_a \hat{\alpha}^m(a)| \geq K^{-1} 2^{-m}$. As the stable manifold of $(\hat{\alpha}^m, 0)$ for $f_{a,0}$ is a perfect parabola, we get the motion of $\mathcal{R}_{\hat{\alpha}^m}$ for $B = 0$. As $(f_{a,B})_{a,B}$ is a C^2-family of map, by taking b small enough we have the requested estimate when B is nonzero.

   Case $n_\alpha > M$. We endow $\partial Y_{\alpha(a_0)}$ with the tubular neighborhood:

$$\phi : \partial^s Y_{\alpha(a_0)} \times \mathbb{R} \to \mathbb{R}^2$$

$$(z, u) \mapsto z + u \chi'(z),$$

For every $a \in w$, every $z \in \partial^s Y_{\alpha(a)}$ is sent by $f^{n_\alpha + 1}$ into $\mathcal{R}_{\alpha^0}(a)$ and so the following equation is satisfied:

$$p_1(f_{a,B}^{n_\alpha + 1}(z) - z') = 0,$$

where z' is the canonical horizontal projection of $f_{a,B}^{n_\alpha + 1}(z)$ into $R_{\alpha^0}(a)$ and $p_1 : \mathbb{R}^2 \to \mathbb{R}$ the vertical projection. In order to use the implicit function theorem, we shall consider the following map:

$$\Psi : (a, z, u) \mapsto p_1(f_{a,B}^{n_\alpha + 1} \circ \phi(z, u) - z'),$$

where z' is the canonical horizontal projection of $f_{a,B}^{n_\alpha+1} \circ \phi(z, u)$ into $R_{\alpha^0}(a)$. We notice that Ψ is a smooth map and that if $z \in \partial^s Y_\alpha(a)$, we have:

$$\Psi(a_0, z, 0) = 0$$

Also by the h-property of (Y_α, χ_α), we have:

$$\|\partial_u(f_{a_0,B}^{n_\alpha+1}) \circ \phi(z, 0)\| \geq e^{\frac{c}{4}n_\alpha}.$$

By the cone property of (Y_α, χ_α), we have:

$$\partial_u(f_{a_0,B}^{n_\alpha+1} \circ \phi)(z, 0) \in \chi_\alpha.$$

Thus the norm of $p_1 \circ \partial_u(f_{a_0,B}^{n_\alpha+1} \circ \phi(z, 0))$ is greater than $\frac{e^{cn_\alpha/4}}{2}$. On the other hand, $\mathcal{R}_{\alpha^0}$ is Kb-C^1-close to $\{y = x - \alpha 60\}$ around z' and $\mathcal{R}_{\alpha^0}$ has bounded derivative in the Banach space of C^1-curves with respect to a. So the projection z' has derivative with respect to u smaller than $K\theta^2\|\partial_u(f_{a_0,B}^{n_\alpha+1} \circ \phi(z, 0))\|$. Therefore:

$$\|\partial_u\Psi(a_0, z, 0)\| \geq \frac{e^{\frac{c}{4}n_\alpha}}{2} - K$$

As $n_\alpha > M$, this enables us to apply the implicit function theorem which provides the existence of a neighborhood V_a of $a \in w$, and a C^2-map:

$$\rho : \ V_a \times \partial^s Y_{\alpha(a_0)} \to \mathbb{R},$$

such that with $\rho_\alpha : \ (a, z) \in V_a \times \partial^s Y_{\alpha(a_0)} \mapsto \phi(z, \rho(a, z)) \in \mathbb{R}^2$, we have:

(i) for every $z \in \partial^s Y_{\alpha(a_0)}$, $\rho(a_0, z) = 0$ and so $\rho_\alpha(a_0, z) = z$,

(iii) $\forall a \in V_a$, $\forall z \in \partial^s Y_{\alpha(a_0)}$,

$$f_{a,B}^{n_\alpha+1}(\rho_\alpha(a, z)) \in R_{\alpha^0}(a)$$

Consequently each curve of $\partial^s Y_{\alpha(a)}$ intersects the image of $\rho_\alpha(a, \cdot)$ at an nonempty open subset.

In order to show (ii), let us compute the derivative $\partial_a \rho$.

(3) $$\partial_a \rho(z, a_0) = -\left[\partial_u\Psi(a_0, z, 0)\right]^{-1} \circ \partial_a\Psi(a_0, z, 0)$$

We already computed a lower bound of $\partial_u\Psi(a_0, z, 0)$. Let us compute an upper bound of $\partial_a\Psi(a_0, z, 0)$. We notice that the map:

$$F : \ (z, a) \in \mathcal{D} \times w \mapsto (f_{a,B}(z), a)$$

has differential of norm less than e^{2c^+}. Also, the derivative of $\alpha^0(a) = \frac{1-\sqrt{1-4a}}{2}$ with respect to a is equal to:

$$\partial_a\alpha^0 = \frac{1}{\sqrt{1-4a}}.$$

For b small, the derivative of $\mathcal{R}_{\alpha^0}(a)$ in the space of C^1-curves with respect to a is bounded by K. Also the angle between the tangent space of $\mathcal{R}_{\alpha^0}$ with the horizontal is close to $\pi/4$. Thus we have:

$$\|\partial_a\Psi(a, z, 0)\| \leq K\|\partial_a f_{a,B}^{n_\alpha+1}(z)\| \leq Ke^{2c^+ n_\alpha}$$

By Equation (3), we get (ii). $\qquad\square$

8.2 Equivalence between $\partial_x f^n$ and $\partial_a f^n$

Proposition 8.2. *Let $f := f_{a,B}$ be k-strongly regular, $S \in \Sigma_k$ and $c_l := c_l(S)$, with $l \leq k$. Let $z \in f_{|Y_{w^M}}^{-M}(Y_{(c_l)})$. Then*

$$\frac{\|\partial_a f_{a,B}^{M+n_{c_l}}\|}{\|\partial_x f^{M+n_{c_l}}\|} = \frac{1}{3} + o(1),$$

where $o(1)$ is small when M is large (and so b is small).

Proof. We prove this proposition by induction on $l \leq k$. We recall that c_1 is the neural piece e. Also by Lemma A.1, $\log(4^M \|\partial_x f^M(z)\|)$ is bounded by K, for $z \in Y_{w^M}$. Let us compute $\partial_a f_{a,B}^n$ for any n. In order to do so, we put: $F : (x, y, a) \mapsto (x^2 + a + y, 0, a) + B(x, y)$. We have:

$$TF := \begin{bmatrix} 2x & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} & & 0 \\ T_{x,y}B & & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

And so an induction gives

$$T_z F^n(\partial_a) = \sum_{j=1}^{n} T_{z_j} f^{n-j}(\partial_x) + \partial_a, \text{ with } z_j := f^j(z), \ \partial_x = (1, 0, 0), \text{ and } \partial_a = (0, 0, 1).$$

In other notations:

$$\partial_a f^n(z) = \sum_{j=1}^{n} \partial_x f^{n-j}(z_j).$$

Let us prove the proposition for $l = 0$. We put $M' := [M/2]$. We have:

$$\partial_a f^M(z) = \sum_{j=1}^{M'} \partial_x f^{M-j}(z_j) + \sum_{j=M'+1}^{M} \partial_x f^{M-j}(z_j)$$

As $\partial_x f^j \leq e^{2jc^+}$ the second term is dominated by $Ke^{2M'c^+}$. On the other hand the M' first iterates of $(z_j)_j$ are very close to $\mathcal{R}_{\alpha^M}$, and so for every $1 \leq j \leq M'$,

$$1 - p_1 \circ \partial_x f(z_j) = 1 + p_1 \circ \partial_x f(z_0) + o(1) = -3 + o(1)$$

This implies that:

$$-\Big(\sum_{j=1}^{M'} \partial_x f^{M-j}(z_j)\Big)(3+o(1)) = (1+o(1))\Big(\sum_{j=2}^{M'} \partial_x f^{M-j}(z_j) \cdot (1-p_1 \circ \partial_x f(z_{j-1})) + \partial_x f^{M-1}(z_1)(1+p_1 \circ \partial_x f(z_0))\Big)$$

By splitting every vector $\partial_x f(z_j)$ following the base (∂_x, e_{M-j}), we get:

$$-\Big(\sum_{j=1}^{M'} \partial_x f^{M-j}(z_j)\Big)(3 + o(1)) = (1 + o(1))\big(\partial_x f^M(z) + \partial_x f^{M-M'}(z_{M'})\big).$$

As $\|\partial_x f^{M-j}(z_j)\| \leq e^{-2cM'} \|\partial_x f^M(z)\|$ for every $j \geq M'$, we get the step $l = 0$ (and also $c_1 = e$).

Let $w := T_z f^M(1,0)$, $w_a^l := TF^{M+n_{c_l}}(\partial_a) - \partial_a$. Let us compare $\|Tf^{n_{c_l}}(w_a^1)\|$ and $\|Tf^{n_{c_l}}(w)\|$ by using the cone field of c_l.

As $c_1 = e$, we have $w_a^1 := TF^M(\partial_a) - \partial_a$. Let us split w_a^1 with respect to the base $(w, e_{n_{c_l}})$:

$$\frac{w_a^1}{\|w_a^1\|} =: uw + ve_{n_{c_l}},$$

with:

$$u := \frac{\sin \angle(w_a^1, e_{n_{c_l}})}{\sin \angle(w, e_{n_{c_l}})\|w\|} \quad \text{and} \quad v := \frac{\sin \angle(w, w_a^1)}{\sin \angle(w, e_{n_{c_l}})}.$$

Since w_a^1 and w belong to the image of χ by Tf or to the cone field of α, and since S_{c_l} is more than $2^{-M}/K$ distant to 0, we have $\angle(w, w_a^1) << \angle(w, e_{n_{c_l}})$. This implies $|v| = o(1)$ and $|u| = \frac{1+o(1)}{\|w\|}$. Therefore

$$\frac{\|Tf^{n_{c_l}}(w_a^1)\|}{\|Tf^{n_{c_l}}(w)\|} = (1 + o(1))\frac{\|w_a^1\|}{\|w\|} = \frac{1}{3} + o(1).$$

Consequently the problem is now to compare $\|Tf^{n_{c_l}}(w_a^1)\|$ and $\|w_a^l\|$.

We have:

$$w_a^l = \sum_{j=1}^{n_{c_l}} T_{z_{j+M}} f^{n_{c_l}-j}(\partial_x) + T_{z_M} f^{n_{c_l}}(w_a^1)$$

We want again to split the vector ∂_x with respect to $Tf^j(w_a^1)$ and $e_{n_{c_l}-j}$. For this end, we shall compare $e_{n_{c_l}-j}$ and e_1. By Proposition 4.4, such a comparison is possible if following projective hyperbolic inequality holds:

$$(4) \qquad \|T_{z'}f^p\| \geq e^{-Mpc^+}, \ \forall p \leq n_{c_l} - j, \ \forall z' \in f^j(Y_{c_l})$$

Let us consider the following family of times:

$$\hat{\tau}_{c_l} := \left\{ j \in \{1, \ldots, n_{c_l}\} : \ \forall l \leq l, \ j \notin \left[n_{c_{l'}} - \frac{n_{\alpha_{l'+1}}}{M}, n_{c_{l'}}\right) \text{ if } n_{\alpha_{l'}} := n_{c_{l'+1}} - n_{c_{l'}} \geq M \right\}$$

We prove below the following Lemma:

Lemma 8.3. *For every $j \in \hat{\tau}_{c_l}$, Inequality (4) holds.*

We notice that as $c_l = \bigstar_{i=1}^l \alpha_i$ is formed mostly by simple pieces, $\frac{Card\tau_{c_l}}{n_{ncl}} \geq 1 - \mu(1 + 1/M)$.

We can split as before ∂_x following the base $\left(T_{z_M}f^j(w_a'), e_{n_{c_l}-j}(z_{M+j})\right)$. This gives that:

$$(5) \qquad \|T_{z_{j+M}}f^{n_{c_l}-j}(\partial_x)\| \leq K\frac{\|T_{z_M}f^{n_{c_l}}(w_a^1)\|}{\|T_{z_M}f^j(w_a^1)\|} \leq Ke^{-cM-cj}\|T_{z_M}f^{n_{c_l}}(w_a^1)\|$$

For $j \notin \hat{\tau}_{c_l}$, let $j' > j$ minimal s.t. j' belongs to τ_{c_l}. We know that $|j' - j| < \mu(1 + 1/M)j'$. The vector $T_{z_{j+M}}f^{j'-j}(\partial_x)$ can be split following the basis $(Tf^{j'}(w_a^1), e_{n_{c_l}-j'})$. Let $(u_j, v_j) \in \mathbb{R}^2$ be such that:

$$T_{z_{j+M}}f^{j'-j}(\partial_x) = u_j T_{z_n}f^{j'}(w_a^1) + v_j e_{n_{c_l}-j}$$

As the angle between $Tf^{j'}(w_a^1)$ and $e_{n_{c_l}-j}$ is greater than $2^{-M}/K$, we have:

$$|u_j| < K2^M \frac{\|T_{z_{j+M}} f^{j'-j}(\partial_x)\|}{\|T f^{j'}(w_a^1)\|}$$

$$|v_j| < K2^M \|T_{z_{j+M}} f^{j'-j}(\partial_x)\|.$$

We have $\|T_{z_{j+M}} f^{j'-j}(\partial_x)\| < e^{2c^+(j'-j)} < e^{2\mu j' c^{++}}$ and $\|T_{z_n} f^{j'}(w_a^1)\| > \frac{e^{c^-(j'+M)}}{K}$.

$$\Rightarrow \|T_{z_{j+M}} f^{n_{c_l}-j}(\partial_x)\| \le K2^M e^{2c^{++}\mu j' - c^-(j'+M)} \|T_{z_n} f^{n_{c_l}}(w_a^1)\| + K2^M b^{(n_{c_l}-j')/2} e^{2(j'j)c^+}$$

(6)
$$\Rightarrow \|T_{z_{j+M}} f^{n_{c_l}-j}(\partial_x)\| \le e^{-\frac{c}{2}(j+M)} \|T_{z_M} f^{n_{c_l}}(w_a^1)\|.$$

Using (5) and (6), we get:

$$\frac{\|w_a^l\|}{\|T f^{n_{c_l}}(w_a^1)\|} \sim 1 + \sum_{j=1}^{n_{c_l}} K e^{-(M+j)c/2} = 1 + o(1).$$

$\square$

Proof of Lemma 8.3. For $j \in \hat{\tau}_{c_l}$, let $i \ge j$ be minimal such that $i = n_{c_{l'}}$ for $l' \le l$. Inequality (4) is clear for $p \le i - j$. Moreover $T_{z'} f^{i-j}(\partial_x)$ belongs to χ. By proceeding as for the common pieces extension (Proposition 4.8), one shows that the piece $\tilde{c} := \alpha_{l'} \star \cdots \star \alpha_l$ is well defined and can be endowed with an extension $(Y_{(\tilde{c})}, \chi_{\tilde{c}})$ s.t. $\partial^u Y_{(\tilde{c})}$ consists of two segments of $\{y = \pm 2\theta\}$ and s.t. $\chi - \tilde{c}$ is the restriction of χ to $Y_{(\tilde{c})}$. Moreover for $(z, u) \in \chi_{\tilde{c}}$, the vector $T_z f^{n_{c_m}-j}(u)$ belongs to the cone field of α_{m+1} for every $m \ge l'$. By proceeding as before, one shows Inequality (4). $\square$

8.3 Strong estimate on the motion of puzzle pieces

Let w be a k-combinatorial component, and let a_0 be in the interior of w. Let $f := f_{a_0,B}$ and $S(a_0)$ be a curve of $\Sigma_k(a_0)$. Let $c_k(a_0)$ be the common piece of depth k associated to S. We want to study the motion of $(c_k \star \alpha)(a)$, for a regular piece $\alpha \in \mathcal{Y}_k$ which is simple or satisfies $n_\alpha \le \mu n_{c_k}$. The piece $c(a) = (c_k \star \alpha)(a)$ should be seen as a candidate for the common sequence of depth $k+1$ associated to S. By Proposition 4.8, $\partial^u Y_{(c)}$ is formed by two segments of the lines $\{y = -2\theta\}$ and $\{y = 2\theta\}$ and its boundary $\partial^s Y_{(c)(a_0)}$ is formed by curves $\sqrt{b}$-C^1-close to arcs of parabola.

We are interested in the dependence on $a \in w$ of the intersection between $Y_{(c)(a)}$ and $f_{a,B}^M(f_{a,B}(S(a)) \cap Y_{w_-^M})$. In the next section, we will study the dependence of $S(a)$ on a. To do not have to do twice the same work, we prefer to study the motion of $f_{a,B|Y_{w_-}}^{-M}(Y_c(a))$ with respect to a and then compare it easily to the one of $f_{a,B}(S(a))$. In order to talk about $[(M+k)/20]$-critical intersection, we notice that the homeomorphism of the square:

$$R_c(a) := \{Y_{w_-^M(a)}, M\} \star \{Y_{c(a)}, n_c\}$$

is well defined, although not attached to a curve and with image outside of Y_e. We notice that $\partial^u R_c^a$ consists of two segments of the lines $\{y = -2\theta\}$ and $\{y = 2\theta\}$. Also $\partial^s R_{c(a)}$ is formed by two curves $\sqrt{b}$-close to arcs of parabolas.

Proposition 8.4. *There exist a neighborhood V_a of a_0 in w and a C^2-function $\rho_\alpha : V_a \times \partial^s R_{c(a_0)} \to \mathbb{R}^2$ such that for every $z \in \partial^s R_{c(a_0)}$:*

(i) $\rho_c(a_0, z) = z$,

(ii) $|\partial_c \rho_\alpha(a, z)| = \frac{1}{3} + o(1)$ *for M large, and $\partial_a \rho_\alpha(a, z)$ is horizontal,*

(iii) $\rho_c(a, \partial^s R_{c(a_0)})$ *is equal to $\partial^s R_{c(a)}$, for every $a \in V_a$.*

Proof. We proceed as we did for the weak estimate. First we endow $\partial^s R_{c(a_0)}$ with the tubular neighborhood:

$$\phi : \partial^s R_{c(a_0)} \times \mathbb{R} \to V$$

$$(z, u) \mapsto z + u \cdot (1, 0),$$

Let χ' be a smooth extension to $\partial^s Y_{\alpha(a_0)}$ of $\partial_x f_{a_0 B}^{n_{c_k}+M}(z)$ for $z \in \partial^s Y_{(c_k)(a_0)}$. As $\chi \subset \chi_\alpha$, we can define the tubular neighborhood:

$$\phi' : \partial^s Y_{\alpha(a_0)} \times U \to V'$$

$$(z, a) \mapsto z + u \chi'(z)$$

which is a diffeomorphism onto its image V' if the neighborhood U of $0 \in \mathbb{R}$ is sufficiently small. We put p_1 and p_2 the compositon of the inverse of ϕ' with the projection on the first and second coordinate respectively.

For every $a \in w$ close to a_0, every $z \in \partial^s R_{c(a_0)}$ is sent by $f_{a,B}^{n_{c_k}}$ into $\partial^s Y_{\alpha(a)}$ and so the following equation is satisfied:

$$p_2(f_{a,B}^{n_c}(z) - z') = 0,$$

with $z' := p_1 \circ f_{a,B}^{n_c+M}(z)$.

In order to use the implicit function theorem, we shall consider the following map:

$$\Psi : (a, z, u) \mapsto p_2(f_{a,B}^{n_\alpha+1} \circ \phi(z, u) - z').$$

We notice that Ψ is a smooth map and that if $z \in \partial^s Y_{\alpha(a_0)}$, we have:

$$\Psi(a_0, z, 0) = 0$$

Also, we have:

$$\|\partial_u(f_{a_0,B}^{n_c+M} \circ \phi)(z, 0)\| \geq e^{c(n_c+2M)}.$$

Thus $Tp_2 \circ \partial_u(f_{a_0,B}^{n_\alpha+1} \circ \phi(z, 0))$ is invertible. This enables us to apply the implicit function theorem which provides the existence of a neighborhood V_a of $a \in w$, and a C^1-map:

$$\rho : V_a \times \partial^s R_{c(a_0)} \to \mathbb{R},$$

such that with $\rho_c : (a, z) \in V_a \times \partial^s R_{c(a_0)} \mapsto z + \rho(a, z)(1, 0) \in \mathbb{R}^2$, we have:

(i) $\rho(a_0, z) = 0$, for every $z \in \partial^s R_{c(a_0)}$,

(ii) $\partial_a \rho_c$ is horizontal,

(iii) $\forall a \in V_a$, $\forall z \in \partial^s R_{c(a_0)}$,

$$f_{a,B}^{n_c+M}(\rho_c(a,z)) \in \partial^s R_{c(a)}.$$

As $\partial^s R_{c(a_0)}$ has its end point in the line $\{y = -2\theta\}$ and $\{y = 2\theta\}$, by (i), (ii) and (iii), the image of $\rho_c(a, \cdot)$ is equal to $\partial^s R_{c(a)}$.

Let us compute the derivative $\partial_a \rho$.

(7) $\partial_a \rho(z, a_0) = -[\partial_u \Psi(a_0, z, 0)]^{-1} \circ \partial_a \Psi(a_0, z, 0) = (Tp_2 \circ \partial_x f_{a_0,B}^{n_c+M})^{-1} Tp_2 [\partial_a f^{n_c+M}(z) - \partial_a z']$

As $\partial_a f^{n_c+M}(z)$ belongs to $\chi_\alpha(a_0)$, by the two last subsections, we get the stated upper bound. $\quad\square$

9 Motion of Σ_k

The following proposition states that the curves of Σ_k vary slowly in the C^1 topology, as a function of a in a k-combinatorial component.

Proposition 9.1. *Let w be a k-combinatorial component and $a_0 \in w$. For $S(a_0) \in \Sigma_k(a_0) \cup \tilde{\Sigma}_k(a_0)$, let $\tilde{S}(a) := \cup_{a \in w} S(a) \times \{a\}$. Then the tangent space of the surface $\tilde{S}$ is included in the cone field $\tilde{\chi}$ on $\mathbb{R}^3$, centered in the plan $\mathbb{R}^2 \times \{0\}$ and with radius equal to 2θ.*

Proof. Let again $F : (x, y, a) \mapsto (x^2 + a + y, 0, a) + B(x, y)$.

Let us prove the proposition for the dense set in Σ_k formed by curve S of the form $S(a) = \mathbb{S}^\alpha$ with α a product of regular pieces of $\mathcal{Y}_k$.

For $\alpha = e$, the proposition is obvious by taking b small. Otherwise, let $n = n_\alpha > 0$.

Given a pair vectors $(v, w) \in \mathbb{R}^3$, we denote by $v \times w$ their vector product. It is orthogonal to the plan spanned by (u, w).

We notice that it is sufficient to prove that for every unit vector u tangent to $S_\alpha(a)$ at some point z:

$$\left| \angle \left(T_{z,a} F^n(\partial'_a) \times T_{z,a} F^n(u), \partial_a \times T_z f^n(u) \right) \right| \leq \theta.$$

with $f := f_{a,B}$ and ∂'_a equal to the sum of $\partial_a = (0, 0, 1)$ with a 2θ-small vector v of $\mathbb{R}^2 \times \{0\}$ such that ∂'_a is tangent to $\tilde{S}^\alpha = \cup_{a \in w} S^\alpha(a) \times \{a\}$ by induction.

We have :

$$T_{z,a} F^n(\partial'_a) \times T_{z,a} F^n(u) = T_{z,a} F^n(\partial_a) \times T_z f^n(u) + T_z f^n(v) \times T_z f^n(u)$$

as the Jacobian of f is b-small, we have:

$$\|T_z f^n(v) \times T_z f^n(u)\| \leq 2\theta b^n$$

We recall that:

$$T_{z,a} F^n(\partial_a) = \sum_{j=1}^{n} T_{z_j} f^{n-j}(\partial_x) + \partial_a,$$

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with $z_j := f^j(z)$.

Thus

$$T_{z,a}F^n(\partial_a) \times T_{z,a}F^n(u) = \sum_{j=1}^{n} T_{z_j}f^{n-j}(\partial_x) \times T_z f^n(u) + \partial_a \times T_z f^n(u)$$

Since the Jacobian of f is less than b, we have:

$$\|T_{z_j}f^{n-j}(\partial_x) \times T_z f^n(u)\| \leq b^{n-j}\|T_z f^j(u)\|$$

By the h-time property, we have:

$$\|T_z f^j(u)\| \leq e^{-c(n-j)/3}\|T_z f^n(u)\| \Rightarrow \|T_{z_j}f^{n-j}(\partial_x) \times T_z f^n(u)\| \leq b^{n-j}e^{-c(n-j)/3}\|T_z f^n(u)\|$$

For $j = n$, we have:

$$\|\partial_x \times T_z f^n(u)\| \leq \theta\|T_z f^n(u)\|$$

Thus

$$\left| \angle\left(T_{z,a}F^n(\partial_a') \times T_z f^n(u), \partial_a \times T_z f^n(u)\right) \right| \leq \frac{\|T_{z,a}F^n(\partial_a' \times u) - \partial_a \times T_z f^n(u)\|}{\|\partial_a \times T_z f^n(u)\|}$$

$$\leq \frac{2\theta b^n + 2\theta + \sum_{j=1}^{n-1} b^{n-j}e^{-c(n-j)/3}}{e^{c\frac{n}{3}}} \leq \theta$$

The proof for the curves of $\tilde{\Sigma}_k$ is done similarly, by induction on k. □

10 Proof of Theorem 0.1

Let us fix M large and then b small, such that everything done before holds. We denote by $\Omega_k = \Omega_k(B)$ the set of parameters a for which $f_{a,B}$ is k-strongly regular, with $M + 1$ as the first critical return time of the quadratic map f_a in $[\alpha^0, \alpha^1]$. We want to evaluate a lower bound of the Lebesgue measure of Ω_k.

A naive idea would be to look at the set of common stable manifolds W_c^s and then pull back them by f^M to make the leaves of a lamination $\mathcal{R}$ which is transversally of Lebesgue positive measure. On the other hand, the lamination Σ is transversally a thick Cantor set, that is of Hausdorff dimension very small. Then $f_{a,B}$ should be strongly regular if all the leaves of Σ have their images by $f_{a,B}$ tangent to leaves of $\mathcal{R}$. When Σ consists of a single leaf, since its image move as fast as ∂_a and since $\mathcal{R}$ move as fast as $\partial_a/3$, we get that the set of strongly regular parameters a should have relative Lebesgue measure positive. Thus the extension to the case where Σ is transversally of very small Hausdorff dimension should be an independent problem.

It is not the case. They move continuously only on each k-combinatorial component.

Let Ω_k^0 be the set of k-combinatorial components w. In order to do not redo the same mistake, we introduce the following notations.

We write $\Sigma_k|w$ (resp. $\tilde{\Sigma}_k|w$, $\check{\Sigma}_k|w$) the set of the homotopies $(S|w) : \; a \in w \mapsto S(a) \in \Sigma_k(a)$ (resp. $\tilde{\Sigma}_k(a)$, $\check{\Sigma}_k(a)$) such that $S(a)$ and $S(a')$ have the same combinatorial definition for all

$a, a' \in w$. Also we write $\mathcal{Y}_k|w$ for the set of the homotopies $(\alpha|w) : a \in w \mapsto \alpha(a) \in \mathcal{Y}_k(a)$ such that $\alpha(a)$ and $\alpha(a')$ have the same combinatorial definition.

For $w' \subset w$, we denote by $(S|w')$ the restriction $(S|w)|w'$ of $(S|w)$ to w' and $(\alpha|w') := (\alpha|w)|w'$. We write $\Sigma_k|w'$ the set $\{(S|w') := (S|w)|w' : (S|w) \in \Sigma_k|w\}$. In particular, if $w' \subset w$ are included in a k'-combinatorial component we have an isomorphism $(S|w) \in \Sigma_k|w \mapsto (S|w') \in \Sigma_{k'}|w'$.

The next naive idea is to suppose that for each component $w \in \Omega_k^0$, each homotopy of curve $S \in \Sigma_k|w \cup \check{\Sigma}_k|w$ composed by $(f_{a,B}^{M+1})_{a \in w}$ has its images that stretches across the common piece $c_k(S)$, then refine a larger subset of this piece in deeper common pieces possible, and estimate of what is lost. Such an idea is difficult since we have many curves S, and we cannot except that this holds for every curve. In the worst scenario, a component w is covered by subintervals $(w_S)_{S \in \Sigma_k|w}$ such that $S(a)$ is in a bad component for every $a \in w_S$. This gives us some trouble to estimate the size of the components of Ω_{k+1} in function of the size of the components of Ω_k. This may happen even if we only consider the curves of $\check{\Sigma}_k$.

This is why we will work with two speeds k and $M^2 k$ along an induction on $k \geq 1$.

For each k-combinatorial component $w \in \Omega_k^0$, we regard $\Sigma_k|w$ and $\check{\Sigma}_k|w$. We recall that $\check{\Sigma}_k|w$ is formed by at most $Ke^{3\mu ck}$ curves $((S^t|w))_t$. To each curve $(S^t|w)$ a family of open subsets $(w_j)_j$ of w is associated. In other words we consider the family of the restrictions $((S^t|w_j))_j$ for a family of open subsets $(w_j)_j$ of w. For each of these intervals w_j, we ask that $(S^t|w_j)$ almost stretches across a common piece $c_{M^2 k}$ of depth $M^2 k$ (Hypothesis H_3 stated below), with $c_{M^2 k}$ made by pieces of $\mathcal{Y}_k$.

Also for every $a \in w_j$, $S^t(a)$ satisfies all the geometric condition of the $M^2 k$-strong regularity, with a little bit more room (H_2). Let $\mathcal{C}_k|w$ be the family $\left((S^t|w_j)\right)_{j,(S^t|w) \in \check{\Sigma}_k|w}$. Let $\mathcal{C}(S^t|w) := \left((S^t|w_j)\right)_j$ for $(S^t|w) \in (\check{\Sigma}_k|w)$.

The point of this construction is to see that the domains $(w_j)_j$ of $\left((S^t|w_j)\right)_j$ are very small compare to w, if there is a homotopy $(S|w)$ in $\Sigma_k|w$ that stretches across a common piece of depth k. Therefore, the union of $(w_j)_j$ is a quiet good approximation of w even if $S|w \neq S^t|w$. As in general such a curve S does not exists since w was defined as a whole k-component, we suppose that each w_j is included in the domain of some homotopy $\mathcal{C}_{\lceil k/M^2 \rceil}(S^{t'}|\hat{w})$ for each $(S^{t'}|\hat{w}) \in \check{\Sigma}_{\lceil k/M^2 \rceil}|\hat{w}$ with $\lceil x \rceil$-the minimal integer greater or equal to a real number x and $\hat{w}$ the $\lceil k/M^2 \rceil$-component containing w (H_1).

Let us state this rigorously.

We construct by induction on k a family of homotopies $\mathcal{C}_k = \{(S|w_j)\}_{\{S,j\}}$ with w_j included in a k-combinatorial component and $(S|w_j) \in \check{\Sigma}_k|w_j$. For every $w \in \Omega_k^0$, every $(S|w) \in \Sigma_k|w$, we denote by $\mathcal{C}_k(S|w)$ the set of the restrictions $(S|w_j)$ that belong to $\mathcal{C}_k$.

When $k = 1$, then $\check{\Sigma}_k|w$ is equal to $(\mathbb{S}|w)$ for the unique 1-combinatorial component $w \in \Omega_0^1$ of 'first return' $M + 1$. The set $\mathcal{C}_1$ is formed by the homotopies $(\mathbb{S}|w_i)$ which satisfy the following properties:

H_1 The domain w_i is contained in the 1-combinatorial component w.

H_2 For every $1 \leq l \leq M^2$ every $a \in w_i$, $\mathbb{S}(a)$ is in a $l' := \left\lceil \frac{M+j}{20} - i \right\rceil$-critical position with a

(common) piece $c_l(a)$ equal to a product of $l-1$ simple pieces:

$$c_l(a) =: \bigstar_{m=2}^l s_m(a).$$

H_3 The domain w_i is a segment $[a^-, a^+]$ such that $f_{a^+,B}^{M+1}(\mathbb{S}(a^+))$ and $f_{a^-,B}^{M+1}(\mathbb{S}(a^-))$ are tangent to $\partial_{1''+}^s Y_{(c_{M^2})}$ and $\partial_{1''-}^s Y_{(c_{M^2})}$ resp. or conversely respectively, with $1'' = \left[\frac{M+M^2}{20} - 1\right]$, and also with $\partial_{l-}^s Y_{(c)} = \partial^s Y_{(c \star \bigstar_1^l s_-^2)(a)} \setminus \partial^s Y_{(c)(a)}$ and $\partial_{l+}^s Y_{(c)} = \partial^s Y_{(c \star s_+^2 \star \bigstar_1^l s_-^2)(a)} \setminus \partial^s Y_{(c)(a)}$ for a common piece c and an integer l.

We notice in H_3 that l' is -1 smaller than it should be canonically following the definition of the l-strong regularity for $l \le M^2$. As every curve of $\Sigma_j(a)$ is close to $\mathbb{S}(a)$ (for any l-regular parameter a), every domain w_i is included in a M^2-combinatorial component.

Let $k > 1$ and let us suppose that every $(S^0|w_i) \in \mathcal{C}_{k-1}$ has its domain w_i included in a k-combinatorial component w. We recall that $(S^0|w_i)$ belongs to $\check{\Sigma}_{k-1}|w_i$.

Let $(S|w_i) \in \check{\Sigma}_k|w_i$ be a successor of $(S_0|w_i)$. This means that for every $a \in w_i$, the curve $S(a)$ is a successor of $S_0(a)$, i.e. $S(a) \succeq S_0(a)$. We consider the family $\mathcal{C}_k(S|w_i)$ of all the subintervals w_{ij} of w_i such that:

H_1 Each $(S'|w_{ij}) \in \check{\Sigma}_{\lceil k/M^2 \rceil}|w_i$ has an extension $(S'|\hat{w}_{ij}(S')) \in \mathcal{C}_{\lceil k/M^2 \rceil}(S')$ whose domain $\hat{w}_{ij}(S')$ contains w_{ij}.

H_2 For every $a \in w_{ij}$, every $l \in (M^2(k-1), M^2 k]$, $S(a)$ is in a $l' := [\frac{M+l}{20} - 1]$-critical position with some common piece $c_l := c_l(S) = \alpha_2(S) \star \cdots \star \alpha_l(S)$ from $\mathbb{S}$, with $(\alpha_l)_{l \ge 2} \in \mathcal{Y}_k$.

H_3 $w_{ij} = [a^-, a^+]$, with $f_{a^+,B}^{M+1}(S(a^+))$ and $f_{a^-,B}^{M+1}(S(a^-))$ are tangent to $\partial_{k''+}^s Y_{c_{M^2 k}}$ and $\partial_{k''-}^s Y_{c_{M^2 k}}$ resp. or conversely resp., with $k'' := \left[\frac{M+M^2 k}{20} - 1\right]$.

We put:

$$\mathcal{C}_k(S|w_i) := \{(S|w_{ij})\}_{ij} \quad \text{and} \quad \mathcal{C}_k := \bigcup \{\mathcal{C}_k(S|w_i) : (S|w_i) \in \check{\Sigma}_k|w_i, \ S \succ S_0 \text{ and } (S_0|w_i) \in \mathcal{C}_{k-1}\}.$$

We notice that H_1 and H_2 imply that for every domain w of homotopy of $\mathcal{C}_k$ is included in a $M^2 k$-combinatorial component: every curve S'' of Σ_l is close to one of $\check{\Sigma}_k$ which is in a $[\frac{M+l}{20} - 1]$-critical position with some c_l, for every $l \le M^2 k$. Thus S'' is in $[\frac{M+l}{20}]$-critical position with the same piece.

In order to estimate a lower bound of the Lebesgue measure of Ω_k, we are going to estimate the measure of the union U_k of the domains of the homotopies of $\mathcal{C}_k$, since U_k is included in $\Omega_{M^2 k}$. We recall that $\cap_k \Omega_k$ is the set of strongly regular parameters a. So the theorem is proved if we show that $\cap_k U_k$ is of relative measure with respect to Ω_1 close to 1, when M is large (and then b small).

We notice that the family of the domains of the homotopies of $\mathcal{C}_k$ is absolutely not disjoint: they overlap as much as the cardinality of $\hat{\Sigma}_k$. Thus to lower bound the measure of $\cap_k U_k$, we are going to compute the proportion of the domains of each homotopy $(S|w)$ that we throw away at each step k. Then we will multiply it by the upper bound $\#_k$ of overlapping.

We put $\mathcal{C}_0 = \{(\mathbb{S}|\Omega_1)\}$. Let $k \geq 1$. We suppose the set $\mathcal{C}_{k-1}$ constructed.

Let $(S'|w_i) \in \mathcal{C}_{k-1}$ and let $(S|w_i) \in \check{\Sigma}_k|w_i$ be a successor of $(S'|w_i)$.

We recall that for every $a \in w_i$, $S(a)$ is in a $\left[\frac{M+k}{20}\right]$-critical position with respect to $c_k(S')(a)$.

In order to satisfy (H_2) and (H_3) at level k for S, we shall split the interval w_i.

Let us regard all the common sequences of pieces $c = (\alpha_i(S))_{i=1}^{kM^2}$ in $\mathcal{Y}_k$ with $\alpha_i = \alpha_i(S)$ for $i \leq M^2(k-1)$, such that there exists $w_i' \subset w_i$ satisfying: $f^{M+1} \circ S|\partial w_i'$ is tangent to $\partial^s Y_c$.

In order to that (H_2) and (H_3) are satisfied we have to take away the slice of w_i' corresponding to

(*) $\alpha_m \star \beta_{\pm}$, with $\alpha_m := c_{(k-1)M^2}(S) \star \alpha_{(k-1)M^2} \star \cdots \star \alpha_m$ satisfying $(k-1)M^2 < m \leq l$ and with $\beta_- := \bigstar_1^{m'} s_-^2$ and $\beta_- := s_+^2 \star \bigstar_1^{m'} s_-^2$.

Let w_c be the remaining slice of w_i'.

Let $\tilde{w}_i$ be the complement of the union of the subsegments $(w_c)_c$ in w_i. We call the connected components of $\tilde{w}_i$ *bad* and throw them away.

We want to estimate the relative measure of the union of the bad components $\tilde{w}_i$ with respect to w_i.

First we can estimate roughly how much we lost when we take away the slice corresponding to (*). The width of $Y_{\bigstar_1^{m'} s_-^2}$ is less than $e^{-2m'c}$. By the distortion property of the extension of α_m, the widths of $\alpha_m \star \beta_{\pm}$ are less than $Ke^{-2m'c}$ the one of α_m and so $Ke^{-2m'c-mc}$ the one of $c_{M^2(k-1)}(S)$. As the motion of $R_c(a)$ with respect to a is 3 times less than the one of $f(S(a))$ for every common piece $c_{(k-1)M^2} \star \alpha_m$, the component w_c corresponding to it has length 3 times greater than the width of $Y_{c_{M^2(k-1)} \star \alpha_m}$ and the bad components corresponding to (*) a width 3 times the widths of $c_{M^2k}(S) \star \alpha_m \star \beta_{\pm}$. Thus we loose a proportion less than to $Ke^{-2m'c-mc}$ of w_c to consider (*) at m. Thus for all m, we lose $\sum_{(k-1)M^2}^{kM^2} Ke^{-2m'c-mc}$. Thus for all w_i we lose at most:

$$v_k := \sum_k e^{-2(k-1)''c^+} |\Omega_1| \cdot \#_k,$$

with $\#_k$ is the upper bound of the cardinality of $\check{\Sigma}_k$ equal to $e^{5\mu k c^+}$, given by Proposition 7.2.

We have also:

$$\sum_{k \geq 1} v_k \leq Ke^{-M/20}.$$

By Proposition 6.20, for every $S' \in \Sigma_i$, the set $\mathcal{E}_k^m(S)$ of the points $z \in S'$ which are not included in a puzzle piece of $\mathcal{Y}_k(S)$ of order less than m is less than $e^{-c\frac{\min(m,k-i)}{4}}$, for every $m, i \leq k$.

On the other hand by Remark 5.5, the curve $\mathbb{S}^{c_j}$ belongs to $\Sigma_{[1+\mu M^3 k]}$ for $j \leq M^2 k$.

Thus the measure of the set of points of $\mathbb{S}^{c_j}$ which are not included in a puzzle piece of $\mathcal{Y}_k$ of order less than m is less than $e^{-c\frac{m}{4}}$, for every $m \leq (1 - \mu M^3)k - 1$.

From this we get that for every $m \geq 0$, the set of points $z \in \mathbb{S}$ which do not belong to a puzzle piece

$$c_{kM^2} := c_{(k-1)M^2}(S) \star \bigstar_{i=(k-1)M^2+1}^{kM^2} \alpha_i,$$

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with $\alpha_i \in \mathcal{Y}_k(\mathbb{S}^{c_{i-1}})$, with $\sum_{\alpha_j \text{ not simple}, j \in [(k-1)M^2+1, kM^2]} n_{\alpha_j}$ less than $m \in [M, (1-\mu M^3)k-1]$ has relative Lebesgue measure in w_i less than:

$$K\left[|\mathcal{E}_k^m| + e^{-2c/3}|\mathcal{E}_k^{m-2}| + \cdots + e^{-2c(m-M)/3}|\mathcal{E}_k^M|\right] \leq \sum_{i=0}^{m-M} K e^{-c(m-i)/4 - ci/3} \leq K e^{-cm/4}.$$

Using the same argument as for the bad components corresponding to $(*)$, one shows that the set of parameters $a \in w_i$ which are not in a critical position with one of the piece above is of relative Lebesgue measure less than $K e^{-cm/4}$.

Proposition 10.1. *The relative measure of $\cup_i \tilde{w}_i$ in Ω_1 is smaller than $u_k + v_k$, where:*

$$\sum_{k \geq 1} u_k + v_k \leq e^{-c/21}$$

Proof. We adapt for our settings the large deviation technique.

For every $(S|w) \in \mathcal{C}_k$, we define on w a sequence $(X_j(S|w))_{M^2(k+1) \geq j \geq 1}$ of integer valued "random function" variables:

For $j \leq M^2 k$ we put:

$$X_j(S|w): \ a \in w \mapsto n_{\alpha_j(S(a))}.$$

If $S(a)$ satisfies H_2 at step $k+1$, we put for $j \in (M^2 k, M^2(k+1)]$:

$$X_j(S|w)(a) := n_{\alpha_j(S(a))}.$$

Otherwise we put $X_{M^2k+1}(S|w)(a) = \max(\mu n_{c_{M^2k}}, M-1) + 2$ and $X_j = 0$ for $j \in [M^2 k + 2, M^2(k+1)]$.

We observe that for every $(S|w) \in \mathcal{C}_k$ and $a \in w$, we have:

$$n_{c_j(S)(a)} = \sum_{l=1}^{j} X_l(S|w)(a),$$

for every $l \leq M^2 k$. Also if $a \in U_k$ but $a \notin U_{k+1}$ because $H_2(k+1)$ failed not because of $(*)$, then there exist $(S|w) \in \mathcal{C}_k$ and $a \in w$ s.t.:

$$\sum_{l=1, X_l \geq M}^{M^2k+1} X_l(S|w)(a) \geq \mu \sum_{l=1}^{M^2k} X_l(a) \geq \mu M^2 k$$

Let $N_j(S|w) := \sum_{l=1}^{j} X_l$ for $j \leq M^2(k+1)$.

We now set, for $(S|w) \in \mathcal{C}_k$ and $l \leq M^2(k+1)$ let:

$$\tilde{X}_l(S|w): \ a \in w \mapsto \begin{cases} 0 & \text{if } X_l(S|w) \leq M \\ X_l(S|w) & \text{if } X_l(S|w) \geq M \end{cases}$$

For $j \leq M^2(k+1)$, put:

$$\tilde{N}_j(S|w) := \sum_{1}^{j} \tilde{X}_l,$$

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We define the sequence of real number $(\phi_k)_k$ by:

$$\Phi_k := \sum_{(S|w)\in\mathcal{C}_k} \int_w e^{\frac{c}{5}\tilde{N}_{(k+1)M^2}(S|w)(a)} da$$

Let us upper bound Φ_k by induction on k. For $(S|w)\in\mathcal{C}_k$, we have:

$$\int_w e^{\frac{c}{5}\tilde{N}_{(k+1)M^2}(S|w)(a)} = \int_w e^{\frac{c}{5}\tilde{N}_{kM^2}(S|w)(a)} e^{\frac{c}{5}\sum_{j=kM^2+1}^{(k+1)M^2} \tilde{X}_j(S|w)(a)} da$$

Also, the function $\tilde{N}_{kM^2}(S|w)$ is constant on its domain w.

$$\int_w e^{\frac{c}{5}\sum_{l=kM^2+1}^{(k+1)M^2} \tilde{X}_l(S|w)(a)} da = \sum_m e^{\frac{c}{5}m} \int_w \mathbb{1}_{\{\sum_{l=kM^2+1}^{(k+1)M^2} \tilde{X}_l(S|w)=m\}} da$$

$$\leq \left(1 + K\sum_{m\geq M} e^{(\frac{c}{5}-\frac{c}{4})m}\right)|w| \leq (1 + Ke^{-\frac{c}{20}M})|w|$$

We conclude that:

$$\int_w e^{\frac{c}{5}\tilde{N}_{(k+1)M^2}(S)(a)} da \leq \int_w e^{\frac{c}{5}\tilde{N}_{kM^2}(S)(a)} da(1 + Ke^{-\frac{cM}{20}})$$

We get that:

$$\Phi_k \leq (1 + Ke^{-\frac{c}{20}M}) \sum_{(S|w)\in\mathcal{C}_k} \int_w e^{\frac{c}{5}\tilde{N}_{kM^2}(S|w)(a)} da$$

For each $(S|w)\in\mathcal{C}_k$ let $(S'|w')\in\mathcal{C}_{k-1}$ be its greatest ancestor: $w'\supset w$ and $S'(a)\preceq S(a)$ for every $a\in w$. By maximality, the curve S' is unique. By Proposition 7.2, the curves $S'(a)$ is $b^{\mu\frac{k-1}{16M}}$ close to $S(a)$ for every $a\in w$. Thus we have $\tilde{N}_{M^2k}(S)(a) = \tilde{N}_{M^2k}(S')(a)$.

For $l\leq k$ and $(S''|w'')\in\mathcal{C}_l$ put:

$$n\big((S''|w''),k\big) := \max_{a\in w''} Card\{(S|w)\in\mathcal{C}_k : a\in w \text{ and } (S''|w'')\preceq(S|w)\}$$

We have:

$$\Phi_k \leq (1 + Ke^{-\frac{c}{20}M}) \sum_{(S'|w')\in\mathcal{C}_{k-1}} n\big((S'|w'),k\big) \int_w e^{\frac{c}{5}\tilde{N}_{M^2k}(S')(a)} da$$

And so on we get:

$$\Phi_k \leq (1 + Ke^{-\frac{c}{20}M^2})^{k-l+1} \sum_{(S|w)\in\mathcal{C}_{l-1}} n((S|w),k) \int_w e^{\frac{c}{5}\tilde{N}_{M^2l}(S|w)(a)} da$$

For $l=2$, we have:

$$\Phi_k \leq (1 + Ke^{-\frac{c}{20}M^2})^{k-1} n\big((\mathbb{S}|\Omega_1),k\big)|\Omega_1| \leq (1 + Ke^{-\frac{c}{20}M^2})^{k-1}\#_k|\Omega_1|,$$

with $\#_k = e^{5\mu k c^+}$.

For $(S|w)\in\mathcal{C}_k$, we have:

$$leb\big(N_{(k+1)M^2}(S|w)\geq k\big) \leq \int_w \frac{e^{\frac{c}{5}N_{(k+1)M^2}(S|w)} - 1}{e^{\frac{c}{5}k} - 1} da,$$

$$\Rightarrow \sum_{(S|w)\in\mathcal{C}_k} leb\{N_{(k+1)M^2}(S|w) \geq k\} \leq \sum_{(S|w)\in\mathcal{C}_k} \int_w \frac{e^{\frac{c}{5}N_{(k+1)M^2}(S|w)} - 1}{e^{kc/5} - 1} da \leq \frac{\Phi_k - \#_k|\Omega_1|}{e^{kc/5} - 1}.$$

$$\Rightarrow u_k \leq \frac{1}{|\Omega_1|} leb\Big\{a : \exists (S|w) \in \mathcal{C}_k, \text{ with } a \in w \text{ and } N_{(k+1)M^2}(S|w)(a) \geq k\Big\} \leq \frac{\Phi_k - \#_k|\Omega_1|}{|\Omega_1|(e^{\frac{c}{5}k} - 1)}$$

$$\leq \#_k \frac{(1 + Ke^{-\frac{c}{20}M^2})^{k-1} - 1}{e^{\frac{c}{5}k} - 1} \leq K \frac{(1 + Ke^{-\frac{c}{20}M^2})^k - 1}{e^{\frac{c}{5}k} - 1} e^{5\mu kc^+}$$

Lemma 10.2. *We have* $\sum_{k\geq 0} u_k \leq Ke^{\frac{-M^2 c^-}{20}}$

The two last estimates with the one of $\sum_{k\geq 1} v_k$ conclude the proof. $\qquad\square$

Proof of Lemma 10.2. We estimate u_k depending on the size of k in the following way:
1) For $k \leq M^2$, we have
$$(1 + Ke^{-\frac{c}{20}M^2})^k - 1 \leq Kke^{-\frac{c}{20}M^2}$$

$$e^{\frac{c}{5}k} - 1 \geq \frac{c}{5}k$$

Hence $u_k \leq Ke^{5\mu kc^+}e^{-\frac{c}{20}M^2}$
2) For $k \geq M^2$,

$$u_k \leq Ke^{5\mu kc^+}\rho^k,$$

with $\rho = \frac{1 + Ke^{-\frac{c}{20}M^2}}{e^{\frac{c}{5}}}$.
Consequently:
$$\sum_{k\geq M^2} u_k \leq K\Big[\frac{1 + Ke^{-\frac{c}{20}M^2}}{e^{\frac{c}{5}}}\Big]^{M^2} e^{4c^+\mu M^2}$$

$$\leq K \cdot e^{-\frac{c}{5}M^2}e^{KM^2 e^{-\frac{c}{20}M^2}} \leq Ke^{-\frac{c}{5}M^2}$$

On the other hand:
$$\sum_{k=1}^{M^2} u_k \leq M^2 Ke^{5\mu M^2 c^+}e^{-\frac{c}{20}M^2} \leq Ke^{-\frac{M^2}{20}c^-}$$

$\qquad\square$

Whereas $(S|w_c)$ satisfies H_2 and H_3, condition H_1 is possibly not satisfied at step $k+1$. That is why we shall take away $(S|w_c)$ if there exists $(S'|w_c) \in \Sigma_{\lceil(k+1)/M^2\rceil}|w_c$ which cannot be extended to an element of $\mathcal{C}\lceil(k+1)/M^2\rceil$. Actually at this step, we do not lose too much measure since the minimal size of such an interval w_c is very small compare to the bad components of the domain w_i of homotopies in $\mathcal{C}_{\lceil(k+1)/M^2\rceil}$.

Lemma 10.3. *For each domain w_i of $\mathcal{C}_{\lceil k/M^2\rceil}$, the length of each of its bad or good component is bigger than r^{-k} times the length of the good one of $\mathcal{C}_k$, with $r := e^{-M^2 c}$.*

Proof. The good components of the domains of the homotopies $(S|w) \in \mathcal{C}_k$ have a size of the same order than the common pieces to which they correspond, that is between $e^{-N_{M^2 k}(S|w)c^+}$ and $e^{-N_{M^2 k}(S|w)c} \leq e^{-2M^2 kc}$.

Similarly, by Proposition 6.20, the components of a curve $S'' \in \Sigma_k$ which are not in the support of a puzzle piece of order less or equal than $m \geq M - 1$, have length greater than: $e^{-2(m+M)c^+}$.

Also the width of $\bigstar_1^{l'-1} s_-^2$ and $s_+^2 \star \bigstar_1^{l'-1} s_-^2$ have length greater than $e^{-2\frac{[}{M+l}20-1]c^+}$.

Thus the bad components of $(S'|w') \in \mathcal{C}_k$ have a length greater than $e^{-2(\mu M^3(k+1)+M)c^+ - N_j(S'|w')c^+ - 2\left[\frac{M+(k+1)M^2}{20}\right]}$ after pullback by $f^{N_j(S'|w')}$ for $j \in [kM^2 + 1, (k+1)M^2]$. Consequently the bad components of $(S'|w') \in \mathcal{C}_{\lceil k/M^2 \rceil}$ have length greater than:

$$e^{-2(\mu Mk + 2\mu M^2) - 2Mc^+ - N_{M^2(1+\lceil k/M^2 \rceil)}(S'|w')c^+}.$$

For $j \leq 2M^2$, $N_j(S|w)$ and $N_j(S'|w')$ are constant function of the the same value, since for b small enough, the $2M^2$-first returns of curves of any curves of $\check{\Sigma}_k$ are the sames.

For $k \geq 2$, the ratio is less than $e^{-kM^2} = r^k$. $\qquad\square$

Let us write by Δ_k the measure of the union of removed components at step $k \geq 2$. We have:

$$\Delta_k \leq u_k + v_k + r^k \Delta_{\lceil k/M^2 \rceil}.$$

$$\Rightarrow \sum_{k \geq 1} \Delta_k \leq \sum_{k \geq 1} (u_k + v_k) + rM^2 \sum_{k \geq 1} \Delta_k.$$

Consequently the relative measure in Ω_1 of the parameters that are not strongly regular is less than:

$$\sum_{k \geq 1} \Delta_k \leq K \sum_{k \geq 1} u_k + v_k \leq e^{-\frac{c}{21}M}$$

The proof of the main theorem is now complete.

$\qquad\square$

A   Computational proof

A.1   Initial Estimate

Here we copy almost words for words the estimates of [Yoc], then we adapt them for the setting of the main theorem. Therefore here we are back to the study of the one dimensional quadratic map $f_a\colon x \mapsto x^2 + a$.

Lemma A.1. *For $M \geq 2$ and any $m \geq 0$ we have:*

$$-\frac{1}{2} \leq \frac{\partial \alpha^m(a)}{\partial a} \leq -\frac{1}{3}.$$

Proof. One has $\alpha^1 = \frac{1}{2}(\sqrt{1-4a}-1)$. Hence $\frac{\partial \alpha^1(a)}{\partial a} = -(1-4a)^{-1/2} \in [-1/2, -1/3]$ if $a \in [-2, -3/4]$.

Then by induction on m, from:

$$(\alpha^m)^2 + a = \alpha^{m-1}$$

we get

$$\frac{\partial \alpha^m(a)}{\partial a} = \frac{\frac{\partial \alpha^m(a)}{\partial a} - 1}{2\alpha^m} < 0$$

Here $\frac{\partial \alpha^m(a)}{\partial a} - 1$ belongs to $[-3/2, -4/3]$ and we have $3/2 \leq \alpha^2 \leq \alpha^m \leq \beta^0 \leq 2$. $\qquad\square$

Let $h_a(x) := (\beta^{0^2} - x^2)^{-1/2}$, for $|x| < \beta^0$. We observe that:

$$h_a(f_a(x)) = [\beta^{0^2} - (x^2 + c)^2]^{-1/2} = [\beta^0 - (x^2+c)]^{-1/2}[\beta^0 + (x^2+c)]^{-1/2} = h_a(x)[x^2 + (\beta^0 + a)]^{-1/2},$$

as $\beta^0 = \beta^{0^2} + c$.

Lemma A.2. *For M large enough we have:*

$$4^{-n} \leq \alpha^n + a \leq K4^{-n}, \quad 0 < n < M - 1$$

$$\frac{2^{-n}}{K} \leq |\hat{\alpha}^n| \leq K2^{-n}, \quad \text{for } 1 < n < M - 1$$

Proof. Using the semi-conjugacy between the angle doubling maps and f_{-2} via $\theta \mapsto 2\cos(\theta)$, for $a = -2$, we have:

$$\alpha^n = -2\cos\frac{\pi}{3 \cdot 2^{n-1}}, \quad \hat{\alpha}^n = -2\sin\frac{\pi}{3 \cdot 2^{n-1}}, \quad c = -\beta_0 = -2$$

Then we use Lemma A.1, which proves that the solution $a^{(n)}$ of the equation $f_a(0) = -\alpha^{(n)}$ in a satisfies:

$$\frac{4^{-n}}{K} \leq a^{(n)} + 2 \leq K4^{-n}$$

This implies the first estimate. The second follows from the first by taking the square roots. $\qquad\square$

We can now use the above estimates and formula to get bounds on some derivatives:

Lemma A.3. *For all $z = (x, 0) \in Y_{s_{\pm}^n}$, $2 \leq n \leq M - 1$, we have:*

$$\left| \log |\partial_x f_a^n(z)| - \log 2^n \cdot \frac{2^n h_a(x)}{h_a(f_a^n(x))} \right| \leq K4^{n-M}$$

And for $z = (x, 0) \in Y_{w_{\pm}^n}$, $n \geq 0$:

$$\left| \log |\partial_x f_a^n(z)| - \log 2^n \cdot \frac{h_a(x)}{h_a(f_a^n(x))} \right| \leq Kn4^{-M}$$

The remaining inequalities of Proposition 2.1 follow easily from the above lemma by taking b sufficiently small with respect to M.

Lemma A.4. *For M large and then b small, for every $z \in Y_{w^M_-}$ we have:*

$$\left| \log \left(4^M \| \partial_x f^M(z) \| \right) \right| \leq K$$

Proof. By Lemma A.2, for b small enough, for all $z = (x, y) \in Y_{w^M_-}$, we have:

$$\left| \log \| \partial_x f^M_a(z) \| - \log 2^M \frac{h_a(x)}{h_a(f^M(x))} \right| \leq K M 4^{-M}$$

with $h_a(x) = (\mathcal{B}^2_0 - x^2)^{-1/2}$. By Lemma A.1, $K 4^{-n} \geq \alpha^n \geq K^{-1} 4^{-n}$ for every $n \geq 0$ and M large. Thus:

$$\frac{2^M}{K} \leq \left| \frac{h_a(x)}{h_a(f^M_a(x))} \right| \leq K 2^M.$$

Thus $K 4^M \leq \left| \log \| \partial_x f^M_a(x) \| \right| \leq K 4^M$. $\qquad\qquad\square$

A.2 Proof of Lemma 6.16

We need the following:

Sublemma A.5. *Let $\sigma' > 0$. Let $(N_i)_i$ and $(N'_i)_i$ be two sequences of matrices of $\mathcal{M}_2(\mathbb{R})$ such that:*

- $\| N_i - N'_i \| \leq \sigma' \cdot \nu^{i-1}$,

- $\| N_i \| \leq K_0$ *and* $\| N'_i \| \leq K_0$.

Then $\| N^{(i)} - N'^{(i)} \| \leq \frac{K_0^{i-1} \sigma'}{1 - \nu}$ *for every $i \geq 1$, with $N^{(i)} := N_i \times \cdots \times N_1$ and $N'^{(i)} := N'_i \times \cdots \times N'_1$*

Proof. Let $\rho_i := \| N^{(i)} - N'^{(i)} \|$. We have:

$$\rho_{i+1} \leq \| N_{i+1}(N^{(i)} - N'^{(i)}) \| + \| (N_{i+1} - N'_{i+1})N'^{(i)} \| \leq K_0 \rho_i + \sigma' \nu^i K_0^i$$

$$\Rightarrow \frac{\rho_{i+1}}{K_0^{i+1}} \leq \frac{\rho_i}{K_0^i} + \sigma' \nu^i K_0^{-1}$$

$$\Rightarrow \frac{\rho_i}{K_0^i} \leq \frac{\sigma'}{K_0} \sum_{j=1}^{i} \nu^{j-1} \leq \frac{\sigma'}{K_0} \cdot \frac{1}{1 - \nu}.$$

$\qquad\qquad\square$

Put $\alpha_i := \| M^{(n_i)} w \times M'^{(n_i)} w' \|$, with $n_i := 2^i$. We have:

$$\alpha_0 := \| M_1 w \times M'_1 w' \| \leq b \| w \times w' \| + \| M_1 w \times (M'_1 - M_1) w' \| \leq b\sigma + K_0 \sigma = (b + K_0)\sigma.$$

We have:

$$\alpha_{i+1} \leq \| N^{(n_i)}(M^{(n_i)} w) \times N^{(n_i)}(M'^{(n_i)} w') \| + \| N^{(n_i)}(M^{(n_i)} w) \times (N^{(n_i)} - N'^{(n_i)}) \cdot M'^{(n_i)} w' \|$$

with $N_j := M_{j+n_i}$, for every $j \geq 0$ and $N^{(n_i)}$ and $N'^{(n_i)}$ as defined in the sublemma.

This implies that $\alpha_{i+1} \leq b^{n_i}\alpha_i + K_0^{3n_i}\sigma'\frac{K_0^{n_i-1}}{1-\nu}$, with $\sigma' := \sigma\nu^{n_i}$.

$$\Rightarrow \frac{\alpha_{i+1}}{\sigma b^{n_{i+1}}} \leq \frac{\alpha_i}{\sigma b^{n_i}} + \frac{\nu^{n_{i+1}/2}K_0^{2n_{i+1}-1}}{(1-\nu)b^{n_{i+1}}} \qquad \Rightarrow \qquad \frac{\alpha_i}{\sigma b^{n_i}} \leq \frac{b+K_0}{b} + \sum_{j=0}^{n_i}\frac{(\frac{\sqrt{\nu}K_0^2}{b})^j}{(1-\nu)K_0}$$

$$\Rightarrow \frac{\alpha_i}{\sigma b^{n_i}} \leq \frac{b+K_0}{b} + \frac{1-(\frac{\sqrt{\nu}K_0^2}{b})^{n_i+1}}{(1-\nu)K_0(1-\frac{\sqrt{\nu}K_0^2}{b})} \qquad \Rightarrow \qquad \frac{\alpha_i}{\sigma b^{n_i}} \leq \frac{2}{K_0}\left(\frac{\sqrt{\nu}}{b}K_0^2\right)^{n_i} + \frac{K_0+b}{b}.$$

$$\Rightarrow \alpha_i \leq \frac{2\sigma}{K_0}\nu^{n_i/2}K_0^{2n_i} + \left(\frac{K_0+b}{b}\right)\sigma b^{n_i}$$

Now let $i \geq 0$ such that $n \in [2^i, 2^{i+1})$.

$$\alpha := \sin\angle(M^{(n)}w, M'^{(n)}w') \leq \|AM^{(n_i)}w \times AM'^{(n_i)}w'\| + \|(A-A')M'^{(n_i)}w' \times M^{(n)}w\|$$

$$\leq b^{n-n_i}\alpha_i + \frac{\sigma\nu^{n_i}}{1-\nu}K_0^{n-n_i-1}K_0^{n_i+n},$$

with $A := M_n \circ \cdots \circ M_{n_i+1}$ and $A' := M'_n \circ \cdots \circ M'_{n_i+1}$.

$$\Rightarrow \alpha \leq \frac{2\sigma}{K_0}\nu^{n/2}K_0^{2n} + \left(\frac{K_0+b}{b}\right)\sigma b^n + \sigma\frac{\nu^{n/2}}{1-\nu}K_0^{2n}$$

$$\Rightarrow \angle(M^{(n)}w, M'^{(n)}w') \leq 2\sigma(K_0^4\nu)^{n/2}$$

A.3   proof of Lemma 6.11

The proof is very similar to Lemma 5.5 of [BC91] (P108), although the statement of the Lemma is slightly different. We show this lemma for n a power of 2. We let the remaining cases to the reader.

For $i \geq 0$, put $n_i = 2^i$, $w_i := M^{(n_i)}(V)$ and $w'_i := M'^{(n_i)}(V)$. Put $u_i := \frac{w_i}{\|w_i\|}$ and $u'_i := \frac{w'_i}{\|w'_i\|}$. Let $A_i := M_{2n_i} \times \cdots \times M_{n_i}$ and $A'_i := M'_{2n_i} \times \cdots \times M'_{n_i}$. We compute:

$$\|w'_{i+1}\| = \|w'_i\|\|A'_iu'_i\| = \|w'_i\|\|A_iu_i + A_i(u'_i - u_i) + (A'_i - A_i)u'_i\|$$

$$\frac{\|w'_{i+1}\|}{\|w'_i\|} \leq \|A_iu_i\| + \|A_i\|\|u'_i - u_i\| + \|A'_i - A_i\|$$

The matrix A_i has norm less than K_0^i. By the previous sublemma $\|A'_i - A_i\| \leq 2\sigma\nu^{n_i-1}K_0^{n_i-1}$. By Lemma 6.16, $\|u'_i - u_i\| \leq 2\sigma(K_0^4\nu)^{n_i/4}$.

$$\Rightarrow \frac{\|w'_{i+1}\|}{\|w'_i\|} \leq \frac{\|w_{i+1}\|}{\|w_i\|} + K_0^{n_i}2\sigma(K_0^4\nu)^{n_i/4} + 2\sigma\nu^{n_i-1}K_0^{n_i-1}.$$

As the vector w_{i+1} as norm greater than 1, $\frac{\|w_i\|}{\|w_{i+1}\|} \leq K_0^{n_i}$.

$$\Rightarrow \frac{\|w'_{i+1}\|}{\|w'_i\|} \leq \frac{\|w_{i+1}\|}{\|w_i\|}\left(1 + K_0^{2n_i}2\sigma(K_0^4\nu)^{n_i/4} + 2\sigma\nu^{n_i-1}K_0^{2n_i-1}\right) < \frac{\|w_{i+1}\|}{\|w_i\|}(1 + 3\sigma r^{n_i}),$$

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with $r := (\nu K_0^{12})^{1/4}$.

$$\Rightarrow \|w_i'\| \leq \|w_i\| \prod_{j=0}^{i} \left(1 + 3\sigma r^{n_i}\right) \leq \exp(\frac{3\sigma r}{1 - 3\sigma r}) \leq 1 + \sigma.$$

Similarly we show that $\|w_i\| \leq \|w_i'\|(1 + \sigma)$.

$\square$

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